

Periodic orbits near a bifurcating slow manifold

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Abstract

This paper studies a class of one-degree-of-freedom Hamiltonian systems with a slowly varying phase. The slowly varying phase is assumed to unfold a Hamiltonian pitchfork bifurcation. The main result of the paper is that there exists an order of $\ln^2 \epsilon^{-1}$ -many periodic orbits that all stay within an $\mathcal{O}(\epsilon^{1/3})$ -distance from the union of the normally elliptic slow manifolds that occur as a result of the bifurcation. As they pass through the bifurcation the time scales are comparable. Here $\epsilon \ll 1$ measures the time scale separation. These periodic orbits are typically “moderately” unstable. This is in contrast with the periodic orbits that remain an $\mathcal{O}(1)$ -distance from the slow manifold. The effect of approaching the normally elliptic slow manifold is therefore to reduce the stability region. The smallest stable orbits that are *persistently* obtained remain further away from the slow manifold being distant by an order $\mathcal{O}(\epsilon^{1/3} \ln^{1/2} \ln \epsilon^{-1})$. The proofs of these statements are based on averaging of two blow-up systems, allowing one to estimate the effect of the singularity, combined with results on asymptotics of the second Painleve equation.

Keywords:

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1. Introduction

Many problems in physics can be reduced to a two degrees of freedom (2-d.o.f.) Hamiltonian system with one d.o.f. being fast relative to another slow d.o.f., see e.g. [2, 5, 12, 18]. The time scale separation is measured by a small parameter $\epsilon \ll 1$. Either time t is such that the velocities of the fast variables are $\mathcal{O}(1)$ while velocities of the slow ones are $\mathcal{O}(\epsilon)$ and the system is said to be fast. Otherwise time $\tau = \epsilon t$ is such that the velocities of the fast variables are $\mathcal{O}(\epsilon^{-1})$ while velocities of the slow ones are $\mathcal{O}(1)$. In this situation the system is said to be slow. To describe the dynamics in these type of systems, one can often apply the theory of adiabatic invariants [1]. To explain this theory, first note that the fast sub-system, obtained from the fast system with $\epsilon = 0$, where the slow variables are fixed as parameters, is an integrable 1-d.o.f. system. Within a region of closed trajectories it is therefore possible to introduce action-angle variables even in the full system for $\epsilon > 0$. Then, by

averaging the Hamiltonian over the fast angle, one obtains a 1-d.o.f. system for the motion of the slow variables with the action appearing as a parameter. This is called the adiabatic approximation. Suppose that the trajectories within the phase plane of slow variables obtained from this approximation are closed. Then the theory says that, in general, the action only perpetually undergoes small oscillations $\mathcal{O}(\epsilon)$ [1, 7]. The phase space is, up to small gaps, filled with invariant tori [1, 6] that are $\mathcal{O}(\epsilon)$ -close to the tori obtained from the adiabatic approximation.

A scenario, relevant to the problem considered here, where such a theory applies is studied in [6]. Here the action-angle variables exist as a result of an elliptic equilibrium within the limiting fast sub-system. Such an equilibrium varies smoothly by the implicit function theorem with respect to the slow variables to form a sub-manifold which is called a normally elliptic *slow manifold* [10]. Moreover, it is assumed that the system, obtained by constraining the fast variables to the slow manifold, has closed trajectories. I will refer to this system as the slow manifold approximation. It agrees with the adiabatic approximation when the action is set to zero.

If there are separatrices on the phase plane of the fast variables described by the fast sub-system, such as in Fig. 1 (b), then the theory of adiabatic invariants needs some further modification, see [11, 14]. In [13, 20] the authors, presented very interesting results for a class of such systems, also including the class of systems considered here (1). They showed that in such systems there is in an order of ϵ^{-1} -many stable periodic orbits that repeatedly move from rotating within the separatrix lobes to rotating outside the separatrix lobes, see Fig. 1 (b). Moreover, there is an order 1 measure of regular motion; something that cannot be observed on Poincaré-sections: The resonance islands are small, being of order ϵ , and are therefore unlikely to be visible on Poincaré sections. Crucial to these arguments was, however, the condition that the crossings occur away from bifurcation points where the time scales are comparable. This condition was realised by taken the action to be greater than $c > 0$, c independent of ϵ . The main result of this paper uncovers what changes when we move close to such bifurcation points and thus investigating the adequateness of the slow manifold approximation in systems with slow manifold bifurcations.

The geometric theory of singular perturbation provides another approach to the description of slow-fast systems. Although this theory is primarily applied to dissipative systems, focus being on normally hyperbolic slow manifolds, the view-point taken here is also relevant to mention in this context.

This theory, also referred to as Fenichel’s theory [3, 4], connects the behaviour of the fast sub-system, where the slow variables are fixed as parameters, with the behaviour of the slow system with $\epsilon = 0$, where the fast variables are constrained to the slow manifold. The connection is made through the slow manifold which the theory says persists the perturbation of $0 < \epsilon \ll 1$. The flow restricted to the slow manifold converges to the flow of the limiting slow system, and the dynamics near the invariant slow manifold is, in some sense, inherited from the limiting fast system. Normally elliptic slow manifolds do not support such a general theory. However, the results of Gelfreich and Lerman in [6] show that, for general 2-d.o.f. analytic Hamiltonian slow-fast systems, the normally elliptic slow manifolds do in some sense also persist, potentially up to small gaps. The invariant slow manifold with gaps are filled with periodic orbits that are $\mathcal{O}(\epsilon)$ -close to the periodic orbits obtained from the slow manifold approximation. One of the aims of this paper is to land somewhere in-between these two different results and approaches, [20] and [6], addressing periodic orbits in systems with separatrix crossing while on the other hand relating this to normally elliptic slow manifolds and the slow manifold approximation.

In [17, 18], I and my co-authors studied different models of tethered satellites. One of these models is a finite-dimensional model that is obtained by replacing the tether connecting the satellite end-points with a spring that goes slack in compression. In [18], we showed for a Galerkin approximation of a more general PDE-model that such “slack spring” model, within this approximation, accurately describes the dynamics. This was a result of us managing to show that the motion remains close to the normally elliptic branches of a bifurcating slow manifold similar to the one shown in Fig. 1 (a). The bifurcation of the slow manifold arose as a result of a pitchfork bifurcation within the limiting fast system; the situation that I am also considering here. We did not explore a more quantitative description of the dynamics near these objects. The main result in this paper, however, applies to the Galerkin model in [18] (upon using the reduction described in Remark 1 below) and thus provide this example with a more detailed description of the dynamics.

The treatment of bifurcating slow manifold has also received attention elsewhere. For example, the references [5, 15, 16] considered the case of sub-critical pitchfork bifurcations in 2-d.o.f. Hamiltonian slow-fast systems. This was motivated by interfaces between ordered and disordered crystalline states. In particular, the references showed the persistence of singular het-

eroclinic solutions connecting equilibria on the normally hyperbolic critical manifold before and after the bifurcation. It was also shown that the heteroclinic connections remain close to the union of the normally hyperbolic branches of the slow manifolds before and after the perturbation. In particular, on the passage through the bifurcation the time scales were comparable. The situation I am addressing here is similar in the sense that I prove the existence of periodic orbits that remain close to the union of the normally *elliptic* branches of the slow manifold.

Summary of main results. Before presenting the problem and stating the main results formally, I will first collect and highlight the results of the paper in words:

- Close to the bifurcating normally elliptic slow manifold there exist many unstable periodic orbits but, if any, only few stable orbits.
- The many unstable periodic orbits are $\mathcal{O}(\epsilon^{1/3})$ -close to the bifurcating normally elliptic slow manifold. When these orbits pass the bifurcation point the time scales are comparable.
- Stable solutions $\mathcal{O}(\epsilon^{1/3})$ -close to the bifurcating normally elliptic slow manifold can always be created (and destroyed) if the small parameter is allowed to vary.
- There are always stable periodic orbits further away from the bifurcating normally elliptic slow manifold at a distance of $\mathcal{O}(\epsilon^{1/3} \ln^{1/2} \ln \epsilon^{-1})$. As these orbit pass the bifurcation point the time scales are *not* comparable.

Problem formulation and main result. I consider the following class of slow-fast Hamiltonian systems:

$$H = v + \frac{1}{2}y^2(1 + x^2M(x^2, y^2, u)) - \frac{1}{2}f(u)x^2 + \frac{1}{2}x^4 + \frac{1}{2}x^6V(x^2, u), \quad (1)$$

$$\omega = dx \wedge dy + \epsilon^{-1}du \wedge dv,$$

which gives rise to the following fast system of equations

$$\begin{aligned} \dot{x} &= y(1 + x^2M + x^2y^2\partial_{y^2}M), \\ \dot{y} &= -x(-f(u) + 2x^2 + x^4(3V + x^2\partial_{x^2}V) + y^2(M + x^2\partial_{x^2}M)), \\ \dot{u} &= \epsilon. \end{aligned} \quad (2)$$

I will motivate this choice further in Remark 1 below. Here $\epsilon \ll 1$, $u \in S^1 = \mathbb{R}/(2\pi\mathbb{Z})$ and M and V are symmetric, smooth functions:

$$\begin{aligned} M &= M(x^2, y^2, u), \\ V &= V(x^2, u). \end{aligned}$$

Also $1 + x^2 M > 0$. The variable v is the conjugate to the slowly varying phase $u \in S^1$ and is introduced merely for later convenience. Moreover, the function $f = f(u)$ is smooth. Together with M and V , it satisfies the following conditions:

(A1) $f(0) = 0 = f(\tau)$ for some $\tau < 2\pi$;

(A2) $f(u) > 0$ for $u \in (0, \tau)$ and $f(u) < 0$ for $u \in (\tau, 2\pi)$;

(A3) f , $M(x^2, y^2, \cdot)$ and $V(x^2, \cdot)$ are symmetric about $u = \tau/2$:

$$f(u) = f(\tau - u), \quad M(x^2, y^2, u) = M(x^2, y^2, \tau - u), \quad V(x^2, u) = V(x^2, \tau - u);$$

(A4) $f'(0) = 1$;

(A5) $\partial_x^2 H|_{x=\kappa(u), y=0} \neq 0$ for all $u \in (0, \tau)$. Here $x^2 = \kappa(u)^2$ solves (4) below.

The assumption (A3) gives rise to a “time-reversible” symmetry \mathcal{T}_τ : If $(x, y, u) = (x, y, u)(t)$ is a solution then so is

$$\mathcal{T}_\tau(x, y, u)(t) \equiv (x, -y, \tau - u)(-t). \quad (3)$$

Given the form of M and V , I have also enforced a further symmetry \mathcal{R} : If $(x, y, u) = (x, y, u)(t)$ is a solution then so is

$$\mathcal{R}(x, y, u)(t) \equiv (-x, -y, u)(t).$$

I will also think of $\mathcal{R} = -Id$ as the reflection acting on (x, y) alone.

The equilibrium $x = 0 = y$ of the fast sub-system, where u appears as a parameter, undergoes a super-critical symmetric pitchfork bifurcation at $u = 0$ and $u = \tau$. Condition (A5) ensures that these are the only bifurcations in the frozen system. Within $u \in (0, \tau)$ there exists two additional equilibria of the form $(x, y) = (\pm\kappa(u), 0)$ with $x^2 = \kappa(u)^2$ solving

$$f(u) - 2x^2 + x^4(3V + x^2\partial_{x^2}V) = 0. \quad (4)$$

Note that $\kappa^2 = \kappa(u)^2$ is smooth as a function of u by the implicit function theorem taking the following form

$$\kappa(u)^2 = \frac{f(u)}{2} (1 + \mathcal{O}(u)), \quad (5)$$

near $u = 0$. I have here used the fact that $f(0) = 0$ (A1). The function $\kappa(u) \neq 0$ for all $u \in (0, \tau)$ cf. (A5). I take $\kappa(u) > 0$, $u \in (0, \tau)$, without loss of generality. I have illustrated the situation in Fig. 1 (a). Here I have also indicated that $x = 0 = y$ is stable/unstable for $u \notin (0, \tau)$ resp. $u \in (0, \tau)$ and $\epsilon = 0$. On the other hand, the equilibria with $x = \pm\kappa(u)$, $y = 0$ only exist for $u \in (0, \tau)$ and are always stable. This gives rise to the following normally elliptic slow manifold

$$M_0 = \left\{ (x, y, u) | y = 0, x = \begin{cases} 0 & u \notin [0, \tau], \\ \pm\kappa(u) & u \in (0, \tau) \end{cases}, u \neq 0, \tau \right\}.$$

Note, however, that it is not uniformly elliptic due to the bifurcations at $u = 0$ or $u = \tau$. Fig. 1 (b) illustrates the dynamics of the limiting fast subsystem in the two regimes highlighting the two separatrix lobe that appear within $u \in (0, \tau)$. I am looking for periodic orbits that remain close to M_0 passing near the bifurcation points at $u = 0$ and $u = \tau$. To explain this differently: The limit $\epsilon = 0$ of the slow system:

$$\begin{aligned} \epsilon x' &= y(1 + x^2 M + x^2 y^2 \partial_{y^2} M), \\ \epsilon y' &= -x(-f(u) + 2x^3 + x^4(3V + x^2 \partial_{x^2} V) + y^2(M + x^2 \partial_{x^2} M)), \\ u' &= 1, \end{aligned}$$

has a singular periodic solution of the form:

$$y_s = 0, |x_s(u)| = \begin{cases} 0 & u \notin (0, \tau), \\ \kappa(u) & u \in (0, \tau), \end{cases} \quad (6)$$

that lies within $\overline{M_0}$. This is the *normally elliptic slow manifold* approximation. There is another singular solution $x = 0 = y$ which enters the *normally hyperbolic* part of the slow manifold. Due to the \mathcal{R} -symmetry this is in fact a true solution for all ϵ and it is cf. e.g. [21] typically highly unstable with multipliers of order $\mathcal{O}(\epsilon^{-1})$. The periodic orbits I am looking for lie close to the singular orbit (6) in the sense that $||x(u)| - |x_s(u)|| + |y(u)|$ is small with respect to ϵ .

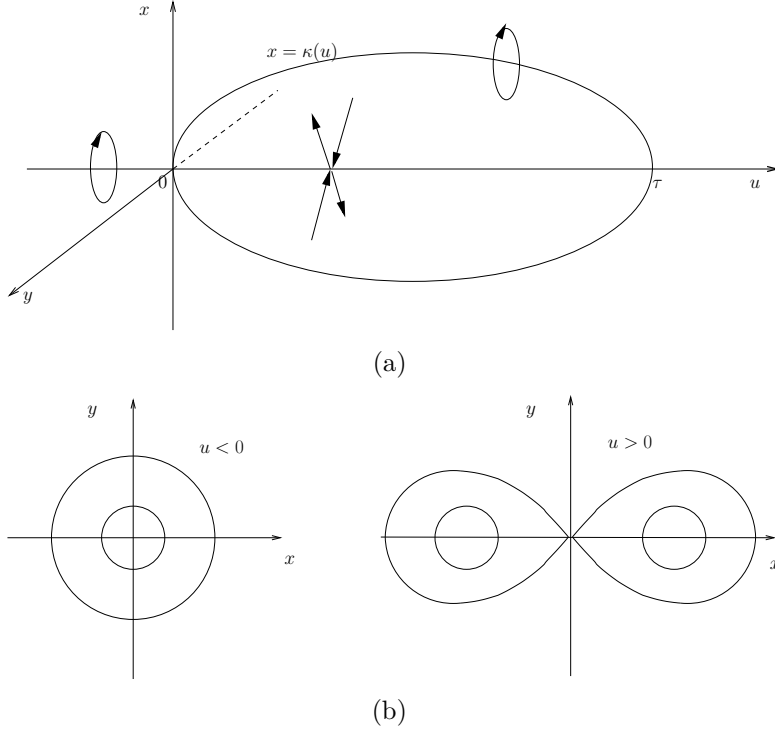


Figure 1: The slow manifold (a) and the dynamics of the frozen system (b). The variables x and y are fast whereas u is slow.

When I later perform some numerical investigations I will base these on the following example

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x(-\sin u + 2x^2), \\ \dot{u} &= \epsilon, \end{aligned} \tag{7}$$

where $f(u) = \sin u$, $M = 0 = V$. In this case $\tau = \pi$.

Remark 1. One of the ways to obtain (2) from a more general setting, is to start from a natural slow-fast 2-d.o.f. system of the form

$$\begin{aligned} K &= \frac{1}{2}(1 + m_f(x^2))y^2 + \frac{1}{2}m_s(w)z^2 + W(w) + x^2Q(x^2, w), \\ \omega &= dx \wedge dy + \epsilon^{-1}dw \wedge dz. \end{aligned}$$

possessing a family of generic periodic orbits within the fix-point set $\{x = 0 = y\}$ of the symmetry action $(x, y) \mapsto \mathcal{R}(x, y) = (-x, -y)$. Assume the following:

- There exists a w_0 so that $Q(0, w_0) = 0$, $\partial_w Q(0, w_0) \partial_{x^2} Q(0, w_0) < 0$;
- The section $\{w = w_0\}$ is transverse to the family of periodic orbits.

One can replace (w, z) by action-angle variables $(I, u) \in \mathbb{R} \times S^1$ within $\{x = 0 = y\}$ and write $K = K(x, y, u, I)$. Locally in (x, y) the system can be reduced by energy solving $K(x, y, u, I) = c$ for I and upon replacing time by $\epsilon^{-1}u$ one can obtain a $1\frac{1}{2}$ -d.o.f. system $I(x, y, u; c)$ that is parametrised by the energy constant c . Up to scalings, translations and $\mathcal{O}(\epsilon)$ -terms, the function $-I$ takes the form (1) with terms satisfying (A1)-(A5). The $\mathcal{O}(\epsilon)$ -terms respect the symmetry and could easily be accounted for but for simplicity of presentation I shall leave out such terms.

The main result of the paper is the following one:

Main result 1. *There exists an ϵ_0 so that the following holds true for all $\epsilon \leq \epsilon_0$:*

- 1° *There exists an order of $\ln^2 \epsilon^{-1}$ -many unstable, slow, and long periodic orbits of (2) where $(x, y) = (x(u), y(u))$ remain $\mathcal{O}(\epsilon^{1/3})$ -close to the union of the normally elliptic critical manifold. Moreover, an order of $\ln^2 \epsilon^{-1}$ of these orbits are symmetric with respect to \mathcal{R} and/or \mathcal{T}_τ . As all of these orbits pass through the bifurcation point at $u \in \{0, \tau\}$ the time-scales are comparable. The characteristic multipliers of the periodic orbits are $\mathcal{O}(\ln^{\pm 2} \epsilon^{-1})$.*
- 2° *For every $\epsilon \leq \epsilon_0$ there are fewer (typically with an order $\ll \ln \epsilon^{-1}$), if any at all, stable periodic orbits of this type than unstable ones.*
- 3° *Take any $\epsilon_1 < \epsilon_0$ and set $I_1 = [\epsilon_1 - c_1 \epsilon_1^2, \epsilon_1 + c_1 \epsilon_1^2] \subset (0, \epsilon_0]$. Then within I_1 there will exist $\lfloor c_2^{-1} \ln \epsilon_1^{-1} \rfloor$ -many closed intervals of lengths $\geq c_3^{-1} \epsilon_1^2 \ln^{-1} \epsilon_1$ for which there exists at least one stable solution $(x, y) = (x(u), y(u))$ remaining $\mathcal{O}(\epsilon^{1/3})$ -close to the union of the normally elliptic critical manifold. Here c_1 , c_2 and c_3 may be large but they can be taken to be independent of ϵ_1 .*

4° *There exist stable, slow, and long periodic orbits, symmetric with respect to \mathcal{R} and/or \mathcal{T}_τ , where $(x, y) = (x(u), y(u))$ remain $\mathcal{O}(\epsilon^{1/3} \ln^{1/2} \ln \epsilon^{-1})$ -close to the union of the normally elliptic critical manifold. As these orbits pass through the bifurcation point at $u \in \{0, \tau\}$ the time-scales are not comparable.*

Remark 2. Regarding 1°: The periodic orbits are *slow* in the sense that the velocities $(\dot{x}, \dot{y}, \dot{u})$ are small. They are *long* in the sense that their periods are large being either $2\pi\epsilon^{-1}$ or $4\pi\epsilon^{-1}$.

Regarding 3°: The solutions within the intervals are distinct but the intervals could potentially overlap giving rise to several solutions for fixed values of ϵ . Following 2°, however, the number of potential overlaps are typically $\ll \ln \epsilon_1^{-1}$. It therefore follows that the relative measure of stable solutions within I_1 is $\gg \ln^{-1} \epsilon^{-1}$. This will also be addressed further in Section 4.6.

Regarding 4°: The orbits in 2° are rare, which we further demonstrate by performing some numerics, and the stable orbits in 4° are, in this sense, *typically* the smallest ones. These orbits are different from those in 1° in that they are stable but their distance to the normally elliptic critical manifold is also larger: $\mathcal{O}(\epsilon^{1/3} \ln^{1/2} \ln \epsilon^{-1})$ vs. $\mathcal{O}(\epsilon^{1/3})$. In contrast to the orbits in 1°, this also means that (\dot{x}, \dot{y}) in this case is in fact larger than \dot{u} when passing through the bifurcation points $u \in \{0, \tau\}$.

Remark 3. Assumption (A3) could be relaxed: It is primarily included by convenience rather than necessity. The problems, such as the one in [18], that arise by the reduction in Remark 1 do, however, satisfy this condition and I therefore found it natural to exploit this.

Remark 4. In the proof of this result it is assumed that $\ln^{-1} \epsilon^{-1} \ll 1$. The constant ϵ_0 , taking the form $\epsilon_0 = e^{-\nu^{-1}}$ with $\nu \ll 1$. The time scale separation therefore need to be “very” large.

The following is a direct consequence of 4°:

Corollary 1. *There exists a family of stable solutions $\{(x(u; \epsilon), y(u; \epsilon)) | 0 < \epsilon \leq \epsilon_0\}$ satisfying $\sup_{u \in S^1} ||x(u; \epsilon) - |x_s(u)|| + |y(u; \epsilon)| = \mathcal{O}(\epsilon^{1/3} \ln^{1/2} \ln \epsilon^{-1})$ for $\epsilon \rightarrow 0^+$.*

Remark 5. Each stable periodic orbit will in general give rise to stability islands. This was also the subject of interest in [20] showing that in the general case there are $\mathcal{O}(\epsilon^{-1})$ -many stability islands of measuring at least $c^{-1}\epsilon$ for some c independent of ϵ . The islands due to the stable periodic orbits in 3° and 4° are expected to be smaller measuring $\mathcal{O}(\epsilon \ln^{-3} \epsilon^{-1})$. I will also discuss this further in Section 5.

Outline. The result is proven by obtaining fix points of a return map

$$\begin{aligned} P : \{(x, y, u) | u = -\pi + \tau/2\} &\rightarrow \{(x, y, u) | u = \pi + \tau/2\}, \\ (x, y, u = -\pi + \tau/2) &\mapsto P(x, y, u) = \phi_{2\pi/\epsilon}(x, y, -\pi + \tau/2), \end{aligned} \quad (8)$$

ϕ_t being the flow of (2). The return map is approximated using averaging and asymptotics of the second Painlevé equation. The averaging principle is applied to two different blow-up systems, one focusing in on $x = 0 = y$ for $u \in (-\pi + \tau/2, -u_*) \cup (\tau + u_*, \pi + \tau/2)$, with u_* small, and one focusing in on $x = \pm\kappa(u), y = 0$ for $u \in (u_*, \tau - u_*)$. Section 2 describes the blow-up transformations used. These blow-ups are appropriate scalings and therefore less complicated than the ones used in e.g. [9] to extend the classical Fenichel theory of slow manifolds near bifurcations. To describe the transition from $u = -u_*$ to $u = u_*$ I make use of the fact that (1), in a certain sense, is close to the second Painlevé equation, where there exists known asymptotics [8]. I also present this asymptotics in Section 2 and show how it can be applied to (1). The small number u_* is written as $\mu^2 \hat{u}_*$ with μ small and connected to ϵ . The number \hat{u}_* is then fixed and $\mathcal{O}(1)$ with respect to ϵ . Connecting the μ with the blow-ups used in the fast space in the two separate regimes, I can obtain a lower bound ($\gg \epsilon$) of the normal frequency for $u \notin (-u_*, u_*) \cup (\tau - u_*, \tau + u_*)$. Such lower bound is crucial to successfully apply the averaging principle. The result, however, cannot be obtained by scalings alone; it is important to make use of the fact that the time-scale separation enhances as u moves away from 0 or τ to be able to accurately approximate the return map. The averaging part is presented in Section 3 where I also describe the return map P in further details. Finally I solve the fix point equations and prove the main result in Section 4.

Notation. I use the convention that the fast variables within the regime $u \in (-\pi + \tau/2, -u_*) \cup (\tau + u_*, \pi + \tau/2)$ are denoted by Roman letters, whereas Greek letters are used within $u \in (0, \tau)$. Blow-up variables are given a hat: $\hat{()}$. Some variables will be given subscripts starting from 0 to indicate that

they later will be updated as a result of transformations. As always in work like this there will be introduced an abundance of constants. I will use c with and without subscripts for constants. They will always be independent of ϵ . To avoid a long enumeration of such constants, I will often restart an enumeration at the beginning of a new section, a lemma, or even a new paragraph. It should be clear from the context where constants are related.

2. Blow-up

In reference [19] they use the following blow-up

$$x = \epsilon^{1/3}\check{x}, \quad y = \epsilon^{2/3}\check{y}, \quad u = \epsilon^{2/3}\check{u}$$

to reduce (2) with $M = 0 = V$ to the Painleve equation of second kind:

$$\begin{aligned} \frac{d\check{x}}{d\check{u}} &= \check{y}, \\ \frac{d\check{y}}{d\check{u}} &= \check{u}\check{x} - 2\check{x}^3, \end{aligned} \tag{9}$$

ignoring here higher order terms that come from the expansion of $f = f(u)$ about $u = 0$. The reference presents asymptotics from [8] of (9) for $\pm\check{u}$ large that I will also make use of here. The asymptotics show that $\check{x} = \mathcal{O}(|\check{u}|^{-1/4})$, $\check{y} = \mathcal{O}(|\check{u}|^{1/4})$ for $\check{u} \rightarrow -\infty$. For $\check{u} \rightarrow \infty$, on the other hand, they show that $\check{x} \mp \sqrt{\hat{u}/2} = \mathcal{O}(\hat{u}^{-1/4})$ and $\check{y} = \mathcal{O}(\hat{u}^{1/4})$. This motivates the following blow-up $\check{x} = \delta^{1/4}\hat{x}$, $\check{y} = \delta^{-1/4}\hat{y}$ when taking $\check{u} = \delta^{-1}\hat{u} < 0$. Then the asymptotics for $u < 0$ can be invoked by letting $\delta \rightarrow 0^+$. For $u > 0$ but small I will use an identical blow-up of the deviation from $\pm\kappa(u) = \pm\sqrt{u/2} + \mathcal{O}(u^{3/2})$.

2.1. Blow-up for $u \in [-(\pi - \tau/2), 0)$

Let $\mu = \epsilon^{1/3}\delta^{-1/2}$. Then motivated by the presentation above I introduce

$$\begin{aligned} x &= \epsilon^{1/3}\delta^{1/4}\hat{x} = \mu\delta^{3/4}\hat{x}, \\ y &= \epsilon^{2/3}\delta^{-1/4}\hat{y} = \mu^2\delta^{3/4}\hat{y}, \\ u &= \epsilon^{2/3}\delta^{-1}\hat{u} = \mu^2\hat{u}, \end{aligned} \tag{10}$$

and insert this into (1):

$$\begin{aligned} H &= v + \mu^4\delta^{3/2} \left(\frac{1}{2}\hat{y}^2 + \frac{1}{2}\hat{F}(\hat{u})^2\hat{x}^2 \right) + \frac{1}{2}\mu^6\delta^3 M_0(u)\hat{x}^2\hat{y}^2 + \frac{1}{2}\mu^4\delta^3\hat{x}^4 \\ &\quad + \mathcal{O}(\mu^{10}\delta^3\hat{x}^2\hat{y}^4 + \mu^6\delta^{9/2}\hat{x}^6), \\ \omega &= \epsilon d\hat{x} \wedge d\hat{y} + \epsilon^{-1/3}\delta d\hat{u} \wedge dv. \end{aligned}$$

I have introduced $M_0 = M(0, 0, u)$ and a scaled frequency \hat{F} defined by

$$\hat{F}(\hat{u})^2 = -\mu^{-2}f(\mu^2\hat{u}) = -\hat{u} + \mathcal{O}(\mu^2). \quad (11)$$

At this stage I think of δ being small, allowing me to invoke the asymptotics of (9), but it cannot be too small as $\mu \ll 1$ so

$$\epsilon \ll \delta.$$

I then introduce $v = \mu^4\delta^{3/2}\hat{v}$ and divide H by $\mu^4\delta^{3/2}$ to obtain a blow-up Hamiltonian system:

$$\begin{aligned} \hat{H} &= \mu^{-4}\delta^{-3/2}H = \hat{v} + \left(\frac{1}{2}\hat{y}^2 + \hat{F}(\hat{u})^2\hat{x}^2\right) + \frac{1}{2}\mu^2\delta^{3/2}M_0(u)\hat{x}^2\hat{y}^2 + \delta^{3/2}\frac{1}{2}\hat{x}^4 \\ &\quad + \mathcal{O}(\mu^6\delta\hat{x}^2\hat{y}^4 + \mu^2\delta^3\hat{x}^6), \\ \hat{\omega} &= \mu^{-4}\delta^{-3/2}\omega = \mu^{-1}d\hat{x} \wedge d\hat{y} + \mu^{-1}\delta^{-3/2}d\hat{u} \wedge d\hat{v}, \end{aligned} \quad (12)$$

giving rise to the following equations of motions

$$\begin{aligned} \dot{\hat{x}} &= \mu(\hat{y} + \mu^2\delta^{3/2}M_0(u)\hat{x}^2\hat{y} + \mathcal{O}(\mu^6\delta^3\hat{x}^2\hat{y}^3 + \mu^2\delta^3\hat{x}^6)), \\ \dot{\hat{y}} &= -\mu\left(\hat{F}(\hat{u})^2\hat{x} + 2\delta^{3/2}\hat{x}^3 + \mu^2\delta^{3/2}\hat{x}\hat{y}^2 + \mathcal{O}(\mu^6\delta^3\hat{x}\hat{y}^4 + \mu^2\delta^3\hat{x}^5)\right), \\ \dot{\hat{u}} &= \mu\delta^{3/2}. \end{aligned}$$

The truncation of the following representation of these equations

$$\begin{aligned} \frac{d\hat{x}}{d\hat{u}} &= \delta^{-3/2}\hat{y} + \mathcal{O}(\mu^2), \\ \frac{d\hat{y}}{d\hat{u}} &= \delta^{-3/2}(\hat{u}\hat{x} - 2\delta^{3/2}\hat{x}^3) + \mathcal{O}(\mu^2\delta^{-3/2}\hat{x}), \end{aligned} \quad (13)$$

coincide with the result of applying the blow-up $\check{x} = \delta^{1/4}\hat{x}$, $\check{y} = \delta^{-1/4}\hat{y}$, $\check{u} = \delta^{-1}\hat{u}$ to the Painleve equations (9) considered in [19].

Action-angle variables. For $u < 0$ I introduce action-angle variables by first setting

$$\hat{x} = \hat{F}(\hat{u})^{-1/2}\hat{x}_0, \quad \hat{y} = \hat{F}(\hat{u})^{1/2}\hat{y}_0.$$

I lift this to a symplectic transformation $(\hat{x}, \hat{y}, \hat{u}, \hat{v}) \mapsto (\hat{x}_0, \hat{y}_0, \hat{u}_0, \hat{v}_0)$ on the full space via the generating function

$$G(\hat{x}, \hat{y}_0, \hat{u}, \hat{v}_0) = \mu^{-1}\delta^{-3/2}\hat{u}\hat{v}_0 + \mu^{-1}\hat{F}(\hat{u})^{1/2}\hat{x}\hat{y}_0,$$

and the equations

$$\begin{aligned}\hat{x}_0 &= \mu \partial_{\hat{y}_0} G, \quad \hat{y} = \mu \partial_{\hat{x}} G, \\ \hat{u}_0 &= \mu \delta^{3/2} \partial_{\hat{v}_0} G = \hat{u}, \quad \hat{v} = \mu \delta^{3/2} \delta \partial_{\hat{u}} G = \hat{v}_0 + \delta^{3/2} \frac{1 + ug(u)}{4\hat{F}(\hat{u})^2} \hat{x}_0 \hat{y}_0,\end{aligned}$$

where $ug(u) = f'(u) - 1 = \mathcal{O}(u)$. Here I have in the last equality used the result from differentiating (11) with respect to \hat{u} . Applying the transformation to (12) gives

$$\begin{aligned}\hat{H} &= \hat{v}_0 + \hat{F}(\hat{u}) \hat{z}_0 + \frac{1}{2} \mu^2 \delta^{3/2} M_0(u) \hat{x}_0^2 \hat{y}_0^2 + \frac{1}{2} \delta^{3/2} \hat{F}(\hat{u})^{-2} \left(\hat{x}_0^4 + \frac{1}{2} (1 + ug(u)) \hat{x}_0 \hat{y}_0 \right) \\ &\quad + \mathcal{O}(\mu^6 \delta^3 \hat{F} \hat{x}_0^2 \hat{y}_0^4 + \mu^2 \delta^3 \hat{F}^{-3} \hat{x}_0^6),\end{aligned}$$

where the action-angle variables (\hat{z}_0, w_0) are the symplectic polar coordinates of (\hat{x}_0, \hat{y}_0) :

$$\hat{x}_0 = \sqrt{2\hat{z}_0} \cos w_0, \quad \hat{y}_0 = \sqrt{2\hat{z}_0} \sin w_0.$$

Remark 6. It will later be shown that the action \hat{z}_0 only undergoes small oscillations. It will follow from this that $x = \mathcal{O}(\epsilon^{1/2})$ when $u \ll 0$. On the other hand when u is such that $\hat{F} = \mathcal{O}(1)$ then $x = \mathcal{O}(\epsilon^{1/3} \delta^{1/4})$.

It will be beneficial to write \hat{H} only in terms of δ and \hat{F} . Consider therefore the term with $\hat{x}_0^2 \hat{y}_0^2$ that also includes μ . I multiply this by $1 = \hat{F}^{-2} \hat{F}^2$ and use the fact that $\mu^2 \hat{F}^2 = -f(u)$ cf. (11). I apply similar reasoning to the remainder. This gives the final form of \hat{H} :

$$\hat{H} = h_0(\hat{u}, \hat{v}, \hat{z}_0) + r_0(\hat{u}, \hat{v}, \hat{z}_0, w_0) + \mathcal{O}(\hat{F}^{-5} \delta^3), \quad (14)$$

splitting it into an integrable part:

$$h_0(\hat{u}, \hat{v}, \hat{z}_0) = \hat{v}_0 + \hat{F}(\hat{u}) \hat{z}_0,$$

and a remainder

$$r_0(\hat{u}, \hat{z}_0, w_0) = \frac{1}{2} \delta^{3/2} \hat{F}(\hat{u})^{-2} \left(\hat{x}_0^4 + \frac{1}{2} (1 + ug(u)) \hat{x}_0 \hat{y}_0 - f(u) M_0(u) \hat{x}_0^2 \hat{y}_0^2 \right).$$

Remark 7. Note how I in these expressions mix u and $\hat{u} = \mu^{-2}u$ together. The reason for introducing \hat{u} is that when $\hat{u} \leq -\hat{u}_* = \mathcal{O}(1)$ then this gives an order 1 lower bound of the scaled frequency:

$$\hat{F}(\hat{u}) \geq \hat{F}(\hat{u}_*),$$

provided μ is sufficiently small. The upper bound $\hat{F}(\hat{u})$ is of order $\mu(\epsilon)^{-1}$ and thus unbounded as $\epsilon \rightarrow 0$. Keeping track of how \hat{F} enters will be crucial when I am to decide what terms are important when I later wish to approximate the solution of these equations by means of averaging. On the other hand, I keep $ug(u)$, for example, in terms of u to highlight that its estimate for u small is not particularly important. Instead it is important to highlight it is smooth and order 1 for all $u \leq 0$.

I will say that big-Oh terms $\mathcal{O}(\hat{F}^{-q}\delta^p)$, like $\mathcal{O}(\hat{F}^{-5}\delta^3)$ in (14), are of order $\hat{F}^{-q}\delta^p$, $q, p > 0$. By this I will mean that they can be bounded *point-wise in* \hat{u} from above by $c\hat{F}(\hat{u})^{-q}\delta^p$, c independent of δ and \hat{u} , for all $\hat{u} \leq -\hat{u}_*$ and δ sufficiently small. I will return later to how different $\hat{F}^{-q}\delta^p$ can be compared.

2.2. Blow-up for $u \in (0, \tau/2]$

To present the asymptotics in [19] for the truncation of (13):

$$\begin{aligned} \frac{d\hat{x}}{d\hat{u}} &= \delta^{-3/2}\hat{y}, \\ \frac{d\hat{y}}{d\hat{u}} &= \delta^{-3/2}(\hat{u}\hat{x} - 2\delta^{3/2}\hat{x}^3), \end{aligned} \tag{15}$$

it is useful to introduce different blow-up variables for $u \in (0, \tau/2)$ which I, as promised, will denote by Greek letters: $(\hat{\xi}, \hat{\sigma})$. They are obtained by performing the blow-up (10) to the deviation $(\pm\xi, \pm\sigma)$ from $(x, y) = (\pm\kappa(u), 0)$:

$$(x, y) = (\pm\kappa(u), 0) + (\pm\xi, \pm\sigma). \tag{16}$$

The particular form is based on the equivariance of the equations with respect to the action of \mathcal{R} . I also introduce symplectic polar coordinates

$$\begin{aligned} \hat{\xi}_0 &\equiv \sqrt{2\hat{\rho}_0} \cos \phi_0 = ((2\hat{u})^{1/4} + \mathcal{O}(\mu\hat{u}^{5/4}))\hat{\xi}, \\ \hat{\sigma}_0 &\equiv \sqrt{2\hat{\rho}_0} \sin \phi_0 = ((2\hat{u})^{-1/4} + \mathcal{O}(\mu\hat{u}^{3/4}))\hat{\sigma}, \end{aligned}$$

much as above. The blow-up Hamiltonian, now denoted by $\hat{\Lambda}$, reads

$$\hat{\Lambda} = \zeta_0(\hat{u}, \hat{\nu}_0, \varrho_0) + \rho_0(\hat{u}, \hat{\varrho}_0, \phi_0) + \mathcal{O}(\hat{\Omega}^{-7/2}\delta^{9/4} + \hat{\Omega}^{-5}\delta^3), \quad (17)$$

$$\hat{\omega} = \mu^{-1}d\hat{\xi}_0 \wedge d\hat{\sigma}_0 + \mu^{-1}\delta^{-3/2}d\hat{u} \wedge d\hat{\nu}_0. \quad (18)$$

splitting $\hat{\Lambda}$ it into an integrable part:

$$\zeta_0(\hat{u}, \hat{\nu}_0, \hat{\varrho}_0) = \hat{\nu}_0 + \hat{\Omega}_0(\hat{u})\hat{\varrho}_0,$$

and a remainder

$$\rho_0(\hat{u}, \hat{\varrho}_0, \phi_0) = \delta^{3/4}\hat{\Omega}(\hat{u})^{-1/2}\rho_{01}(\hat{u}, \hat{\varrho}_0, \phi_0) + \delta^{3/2}\hat{\Omega}(\hat{u})^{-2}\rho_{02}(\hat{u}, \hat{\varrho}_0, \phi_0),$$

where

$$\rho_{01}(\hat{u}, \hat{\varrho}_0, \phi_0) = (1 + \mathcal{O}(u))\hat{\xi}_0^3 - \frac{1}{2}(1 + \mathcal{O}(u))\hat{\sigma}_0 + uM_0(u)(1 + \mathcal{O}(u))\hat{\xi}_0\hat{\sigma}_0^2, \quad (19)$$

$$\begin{aligned} \rho_{02}(\hat{u}, \hat{\varrho}_0, \phi_0) &= \frac{1}{2}(1 + \mathcal{O}(u))\hat{\xi}_0^4 + (1 + \mathcal{O}(u))\hat{\xi}_0\hat{\sigma}_0 + u^3M_{02}(u)(1 + \mathcal{O}(u))\hat{\sigma}_0^4 \\ &\quad + uM_0(u)(1 + \mathcal{O}(u))\hat{\xi}_0^2\hat{\sigma}_0^2. \end{aligned} \quad (20)$$

Here

$$\hat{\Omega}(\hat{u})^2 = 2\mu^{-2}\vartheta(u) = 2\hat{u} + \mathcal{O}(\mu^2\hat{u}^2), \quad (21)$$

is a scaled frequency with

$$\vartheta(u) = 2\kappa(u)^2(1 + 2\kappa(u)^2V_0(u) + \kappa(u)^4V_{01}(u))(1 + \kappa(u)^2M_0(u))^{-1} = u + \mathcal{O}(u^2), \quad (22)$$

$M_{02} = \partial_{y^2}M(\kappa(u)^2, 0, u)$, $V_0 = V(\kappa(u)^2, u)$ and $V_{01} = \partial_{x^2}V(\kappa(u)^2, u)$. From (21) also follows

$$\hat{\Omega}'(u) = \frac{1 + \mathcal{O}(u)}{\hat{\Omega}}, \quad (23)$$

$$\mu^2\hat{\Omega}^2 = 2u(1 + \mathcal{O}(u)), \quad (24)$$

both of which I will be using later on. The details of the $\mathcal{O}(u)$ -terms are not important here. It is just to important to highlight that they are uniform up until $u = 0$. Further details of the derivation of (17) are available in App. Appendix A.

Remark 8. It will later be shown that the action $\hat{\rho}_0$ only undergoes small oscillations. It will from this follow that $x \mp \kappa(u) = \mathcal{O}(\epsilon^{1/2})$ when $u \gg 0$. On the other hand when u is such that $\hat{\Omega} = \mathcal{O}(1)$ then $x \mp \kappa(u) = \mathcal{O}(\epsilon^{1/3}\delta^{1/4})$.

Having now introduced both (\hat{z}_0, w_0) and $(\hat{\rho}_0, \phi_0)$, I am ready to present the asymptotics from [19] that I need.

Lemma 1. *Consider (15) and fix $c > 0$ large. Assume moreover that $\hat{z}_0 \geq c^{-1}$ and that $\lambda = \lambda(\hat{z}_0, w_0)$ given by (33) belongs to the interval $[c^{-1}, \pi - c^{-1}] \cup [\pi + c^{-1}, 2\pi - c^{-1}]$. Then for $\hat{u} \geq \delta\check{u}_*$ with \check{u}_* sufficiently large I have the following asymptotics:*

$$\begin{aligned}\hat{x}_0(-\hat{u}) &= \sqrt{2\hat{z}_0} \cos w_0, \\ \hat{y}_0(-\hat{u}) &= \sqrt{2\hat{z}_0} \sin w_0,\end{aligned}\tag{25}$$

and

$$\begin{aligned}\hat{\xi}_0(\hat{u}) &= \sqrt{2\hat{\rho}_0} \cos \phi_0, \\ \hat{\sigma}_0(\hat{u}) &= \sqrt{2\hat{\rho}_0} \sin \phi_0,\end{aligned}\tag{26}$$

with the action-angle variables (\hat{z}_0, w_0) and $(\hat{\rho}_0, \phi_0)$ related by the following expressions

$$w_0 = \frac{2}{3}\delta^{-3/2}\hat{u}^{3/2} + \frac{3}{2}\hat{z}_0 \ln(\delta^{-1}\hat{u}) - \frac{\pi}{2} + l,\tag{27}$$

$$\hat{\rho}_0 = \frac{1}{2\pi} \ln \frac{1 + |p|^2}{2|\operatorname{Im} p|},\tag{28}$$

$$\phi_0 = -\frac{2\sqrt{2}}{3}\delta^{-3/2}\hat{u}^{3/2} + 3\hat{\rho}_0 \ln(\delta^{-1}\hat{u}) - \theta,\tag{29}$$

where

$$\theta(\hat{\rho}_0, p) = Q(\hat{\rho}_0) - \arg(1 + p^2),\tag{30}$$

$$Q(\hat{\rho}_0) = -\frac{\pi}{4} + 7\hat{\rho}_0 \ln 2 - \arg \Gamma(2i\hat{\rho}_0),\tag{31}$$

$$p(\hat{z}_0, \lambda) = (e^{2\pi\hat{z}_0} - 1)^{1/2} e^{i\lambda},\tag{32}$$

$$\lambda = 3\hat{z}_0 \ln 2 - \frac{\pi}{4} - \arg \Gamma(i\hat{z}_0) - l.\tag{33}$$

If λ in (33) belongs to $[\pi + c^{-1}, 2\pi - c^{-1}]$ then the $+$ sign should be taken in (16). If $\lambda \in [c^{-1}, \pi - c^{-1}]$ then the $-$ sign should be taken (16).

Moreover, if $\hat{u} \geq \hat{u}_*$, with \hat{u}_* fixed, then the errors in (25) and (26) are $\delta^{3/2}$ resp. $\delta^{3/4}$.

PROOF. I use Eqs. (3)-(7) in [19] and write them in the blow-up variables (\hat{x}_0, \hat{y}_0) resp. $(\hat{\xi}_0, \hat{\sigma}_0)$. Their s is my $\check{u} = \delta^{-1}\hat{u}$. Moreover, their α and ρ are related to my \hat{z}_0 and $\hat{\varrho}_0$ by $\alpha^2 = 2\hat{z}_0$ and $\rho^2 = 2\hat{\varrho}_0$. To relate their ϕ and θ to my w_0 and ϕ_0 I also end up having to solve the equations

$$\begin{aligned}\cos w_0 &= \sin \left(\frac{2}{3} \delta^{-3/2} \hat{u}^{3/2} + \frac{3}{2} \hat{z}_0 \ln(\delta^{-1} \hat{u}) + l \right), \\ \sin w_0 &= -\cos \left(\frac{2}{3} \delta^{-3/2} \hat{u}^{3/2} + \frac{3}{2} \hat{z}_0 \ln(\delta^{-1} \hat{u}) + l \right),\end{aligned}$$

and

$$\begin{aligned}\cos \phi_0 &= \cos \left(\frac{2\sqrt{2}}{3} \delta^{-3/2} (2\hat{u})^{3/2} - 3\hat{\varrho}_0 \ln(\delta^{-1} \hat{u}) + \theta \right), \\ \sin \phi_0 &= -\sin \left(\frac{2\sqrt{2}}{3} \delta^{-3/2} (2\hat{u})^{3/2} - 3\hat{\varrho}_0 \ln(\delta^{-1} \hat{u}) + \theta \right),\end{aligned}$$

with respect to w_0 and ϕ_0 . The solutions are

$$w_0 = \frac{2}{3} \delta^{-3/2} \hat{u}^{3/2} + \frac{3}{2} \hat{z}_0 \ln(\delta^{-1} \hat{u}) - \frac{\pi}{2} + l, \quad (34)$$

$$\phi_0 = -\frac{2\sqrt{2}}{3} \delta^{-3/2} \hat{u} + 3\hat{\varrho}_0 \ln(\delta^{-1} \hat{u}) - \theta. \quad (35)$$

The assignment $(\hat{x}, \hat{y})(-\hat{u}) \mapsto (\hat{\xi}, \hat{\sigma})(\hat{u})$, $\hat{u} \geq \delta \check{u}_*$, in Lemma 1 is two-to-one due its invariance with respect to the symmetry $(\hat{x}, \hat{y}) \mapsto \mathcal{R}(\hat{x}, \hat{y}) = (-\hat{x}, -\hat{y})$; in the action-angle variables (\hat{z}_0, w_0) the symmetry corresponds to a translation, which we continue to denote by \mathcal{R} , of w_0 by π : The pair $(\hat{\varrho}_0, \phi_0)$ in (28) and (29) is invariant with respect to this shift. However, if I further assign the sign in (16) to the image of this assignment then I obtain a one-to-one mapping. Therefore consider $U = [c_1, c_2] \times S^1 \subset \{(\hat{z}_0, w_0)\}$ and remove the “bad set” from this set:

$$V = U \setminus \{(\hat{z}_0, w_0) \in U \mid \lambda(\hat{z}_0, w_0) \in [-c^{-1}, c^{-1}] \cup [-c^{-1} + \pi, \pi + c^{-1}]\}. \quad (36)$$

Note that V is open and large in measure, the complement having a measure of order c^{-1} with c large. Indeed, the set V is strip-like, the strips that are excluded from U are closed sets having widths of order $c^{-1} \ln^{-1} \delta^{-1}$ cf. (29). The strips that are included in V , on the other hand, are open with non-empty interior having a width of order $\ln^{-1} \delta^{-1}$. There are $\ln \delta^{-1}$ of such strips. Then

Definition 1. for any $(\hat{z}_0, w_0) \in V$ I introduce P_{cr} and $P_{\text{cr}}^{\text{ext}}$ by $P_{\text{cr}}^{\text{ext}}(\hat{z}_0, w_0) = (\hat{\varrho}_0, \phi_0, \eta)$ where the pair $(\hat{\varrho}_0, \phi_0) = P_{\text{cr}}(\hat{z}_0, w_0)$ is the image of P_{cr} and given by (28) and (29). The argument $\eta \in \{\pm 1\}$ is $\eta = +1$ when $\lambda \in [\pi + c^{-1}, 2\pi - c^{-1}]$ and $\eta = -1$ when $\lambda \in [c^{-1}, \pi - c^{-1}]$.

The subscript cr in P_{cr} and $P_{\text{cr}}^{\text{ext}}$ is for “crossing.” The superscript ext is for “extended.” I have:

Lemma 2. *The assignment $P_{\text{cr}}^{\text{ext}} : V \rightarrow P_{\text{cr}}^{\text{ext}}(V)$ is a diffeomorphism. Moreover, it commutes with the translation $(\hat{z}_0, w_0) \mapsto \mathcal{R}(\hat{z}_0, w_0) = (\hat{z}_0, w_0 + \pi)$ in the following way:*

$$P_{\text{cr}}(\mathcal{R}(\hat{z}_0, w_0)) = P_{\text{cr}}(\hat{z}_0, w_0), \quad \eta(\mathcal{R}(\hat{z}_0, w_0)) = -\eta(\hat{z}_0, w_0).$$

PROOF. For (15) the result follows from the asymptotics in Lemma 1. The fact that the conclusion does not change, when the remainder in (13) is included, is the subject of the following section.

2.3. Applying the asymptotics of the second Painleve Eq. to Eq. (13)

So far the asymptotics above is only valid for the truncation (15) of our blow-up (13). The remainder in (13) that is ignored by the truncation is of order $\mu^2 \delta^{-3/2} |\hat{x}|$ to lowest order. To control the remainder I first need to estimate the truncation’s growth with respect to δ .

Corollary 2. *Suppose that the assumption of Lemma 1 holds true and consider $\hat{u} \in [-\hat{u}_*, \hat{u}_*]$. Then for δ sufficiently small there exists a constant c so that the assignment $\Psi_{\hat{u}} : (\hat{x}, \hat{y})(-\hat{u}_*) \mapsto (\hat{x}, \hat{y})(\hat{u})$ has the following growth properties with respect to δ :*

$$\begin{aligned} |\hat{x}(\hat{u})| &\leq c\delta^{-1/4}, |\hat{y}(\hat{u})| \leq c\delta^{1/4}, \quad 0 \leq \hat{u} < \check{u}_*\delta, \\ |\hat{x}(\hat{u})| &\leq c\delta^{-3/4}\hat{u}^{1/2}, |\hat{y}(\hat{u})| \leq c\hat{u}^{1/4}, \quad \hat{u} \geq \check{u}_*\delta. \end{aligned}$$

Here \check{u}_* is from Lemma 1.

Moreover, for $\hat{u} = \hat{u}_*$ the Jacobian of this assignment satisfies

$$|\partial\Psi_{\hat{u}_*}|, |\partial\Psi_{\hat{u}_*}^{-1}| \leq c \ln^2 \delta^{-1}. \quad (37)$$

For any $\hat{u} \in (-\hat{u}_*, \hat{u}_*)$ the more pessimistic estimate applies:

$$|\partial\Psi_{\hat{u}}|, |\partial\Psi_{\hat{u}}^{-1}| \leq c\delta^{-1/4} \ln \delta^{-1}. \quad (38)$$

PROOF. I focus on the estimates for the Jacobian. The first statement about the growth of \hat{x} and \hat{y} will follow from similar estimates. First I take $\hat{u} \leq -\check{u}_*\delta$ with \check{u}_* large as in Lemma 1. Then from the results presented in that lemma I obtain the following asymptotics

$$\begin{aligned} \partial_{(\hat{x}(-\hat{u}_*), \hat{y}(-\hat{u}_*))} \begin{pmatrix} \hat{x}(\hat{u}) \\ \hat{y}(\hat{u}) \end{pmatrix} &= \partial_{(\hat{z}_0, w_0)} \begin{pmatrix} \hat{x}(\hat{u}) \\ \hat{y}(\hat{u}) \end{pmatrix} \partial_{(\hat{z}_0, w_0)} \begin{pmatrix} \hat{x}(-\hat{u}_*) \\ \hat{y}(-\hat{u}_*) \end{pmatrix}^{-1} \\ &= \mathcal{O} \left(\ln \delta^{-1} \begin{pmatrix} \delta^{-1/4} & \delta^{-1/4} \\ \delta^{1/4} & \delta^{1/4} \end{pmatrix} \right), \end{aligned}$$

For $\hat{u} \in (-\check{u}_*\delta, \check{u}_*\delta)$ I use the coordinates: $(\check{x}, \check{y}) = (\delta^{1/4}\hat{x}, \delta^{-1/4}\hat{y})$ and \hat{u} replaced by $\check{u} = \delta^{-1}\hat{u}$, also used in [19], to control the assignment $(\check{x}, \check{y})(-\check{u}_*) \mapsto (\check{x}, \check{y})(\hat{u}\delta^{-1})$, with a bound that is independent of δ . Returning to my coordinates I then obtain the following

$$\partial_{(\hat{x}(-\check{u}_*\delta), \hat{y}(-\check{u}_*\delta))} \begin{pmatrix} \hat{x}(\hat{u}) \\ \hat{y}(\hat{u}) \end{pmatrix} = \mathcal{O} \begin{pmatrix} 1 & \delta^{-1/2} \\ \delta^{1/2} & 1 \end{pmatrix}.$$

Therefore

$$\begin{aligned} \partial_{(\hat{x}(-\hat{u}_*), \hat{y}(-\hat{u}_*))} \begin{pmatrix} \hat{x}(\hat{u}) \\ \hat{y}(\hat{u}) \end{pmatrix} &= \partial_{(\hat{x}(-\check{u}_*\delta), \hat{y}(-\check{u}_*\delta))} \begin{pmatrix} \hat{x}(\hat{u}) \\ \hat{y}(\hat{u}) \end{pmatrix} \partial_{(\hat{x}(-\hat{u}_*), \hat{y}(-\hat{u}_*))} \begin{pmatrix} \hat{x}(-\check{u}_*\delta) \\ \hat{y}(-\check{u}_*\delta) \end{pmatrix} \\ &= \mathcal{O} \left(\ln \delta^{-1} \begin{pmatrix} 1 & \delta^{-1/2} \\ \delta^{1/2} & 1 \end{pmatrix} \begin{pmatrix} \delta^{-1/4} & \delta^{-1/4} \\ \delta^{1/4} & \delta^{1/4} \end{pmatrix} \right) \\ &= \mathcal{O} \left(\ln \delta^{-1} \begin{pmatrix} \delta^{-1/4} & \delta^{-1/4} \\ \delta^{1/4} & \delta^{1/4} \end{pmatrix} \right). \end{aligned}$$

Finally, I consider $\hat{u} \geq \check{u}_*\delta$ and use the following

$$\begin{aligned} \partial_{(\hat{x}(-\hat{u}_*), \hat{y}(-\hat{u}_*))} \begin{pmatrix} \hat{x}(\hat{u}) \\ \hat{y}(\hat{u}) \end{pmatrix} &= \partial_{(\hat{\rho}_0, \phi_0)} \begin{pmatrix} \hat{x}(\hat{u}) \\ \hat{y}(\hat{u}) \end{pmatrix} \partial_{(\hat{z}_0, w_0)} \begin{pmatrix} \hat{\rho}_0 \\ \phi_0 \end{pmatrix} \partial_{(\hat{z}_0, w_0)} \begin{pmatrix} \hat{x}(-\hat{u}_*) \\ \hat{y}(-\hat{u}_*) \end{pmatrix}^{-1} \\ &= \mathcal{O} \left(\ln \delta^{-1} \begin{pmatrix} \hat{u}^{-1/4} \ln(\delta^{-1}\hat{u}) & \hat{u}^{-1/4} \ln(\delta^{-1}\hat{u}) \\ \hat{u}^{1/4} \ln(\delta^{-1}\hat{u}) & \hat{u}^{1/4} \ln(\delta^{-1}\hat{u}) \end{pmatrix} \right). \end{aligned}$$

Setting $\hat{u} = \hat{u}_*$ here gives (37). For (38) I combine the asymptotics to obtain the following bound

$$\left| \partial_{(\hat{x}(-\hat{u}_*), \hat{y}(-\hat{u}_*))} \begin{pmatrix} \hat{x}(\hat{u}) \\ \hat{y}(\hat{u}) \end{pmatrix} \right| \leq c\delta^{-1/4} \ln \delta^{-1},$$

uniform in \hat{u} . The estimate of the inverses can be derived in a similar manner using the fact that the Jacobian has determinant equal to 1.

Using this lemma I can then describe how small ϵ should be relative to δ to be able to apply Lemma 1 to our blow-up (13).

Lemma 3. *Lemma 1 applies to (13) too provided*

$$\epsilon^{2/3} \delta^{-7/2} \ln^3 \delta^{-1} \ll 1. \quad (39)$$

PROOF. Denote by $z = (\hat{x}, \hat{y})$ the solution of (13). Then I set

$$z(\hat{u}) = \Psi_{\hat{u}}(\Delta(\hat{u})), \quad (40)$$

with $\Delta(\hat{u})$ unknown and $\Psi_{\hat{u}}$ the flow-map from Corollary 2 associated with (15) with $\Psi_{-\hat{u}_*} = Id$. This last equality also implies that $z(-\hat{u}_*) = \Delta(-\hat{u}_*)$. By differentiating (40) with respect to \hat{u} I obtain an equation for $\Delta = \Delta(\hat{u})$:

$$\partial \Psi_{\hat{u}} \Delta'(\hat{u}) = \mathcal{O}(\mu^2 \delta^{-9/4}).$$

I have here used Corollary 2 to conclude that for every \hat{u} the remainder of order $\mu^2 \delta^{-3/2} |\hat{x}|$ is bounded by a term of order $\mu^2 \delta^{-9/4}$. I then use (37) to invert $\partial \Psi_{\hat{u}}$ and integrate the resulting equation from $-\hat{u}_*$ to \hat{u}_* to obtain

$$\Delta(\hat{u}_*) = \Delta(-\hat{u}_*) + \mathcal{O}(\mu^2 \delta^{-5/2} \ln \delta^{-1}) = z(-\hat{u}_*) + \mathcal{O}(\mu^2 \delta^{-5/2} \ln \delta^{-1}).$$

Inserting this into (40) finally gives:

$$z(\hat{u}_*) = \Psi_{\hat{u}_*}(z(-\hat{u}_*)) + \mathcal{O}(\mu^2 \delta^{-5/2} \ln^3 \delta^{-1}),$$

where have also used (38) to expand $\Psi_{u_*}(\Delta(\hat{u}))$. Inserting $\mu^2 = \epsilon^{2/3} \delta^{-1}$ completes the proof.

Note that Lemma 2 also follows provided (39) is valid.

3. The return map

It is natural to view the return map P (8) as a stroboscopic mapping that assigns initial conditions (\hat{z}_0, w_0) at $u = -\pi + \tau/2$ to final conditions 2π -later at $u = \pi + \tau/2$. That is

$$(\hat{z}_0, w_0) \mapsto P(\hat{z}_0, w_0) = \phi_{2\pi}(\hat{z}_0, w_0; -\pi + \tau/2),$$

with $\phi_u(\hat{z}_0, w_0; u_0)$, satisfying $\phi_{u_0}(\hat{z}_0, w_0; u_0) = (\hat{z}_0, w_0)$, being the flow of the non-autonomous system (13). For simplicity I will write this as

$$(\hat{z}_0, w_0)(u = -\pi + \tau/2) \mapsto P(\hat{z}_0, w_0) = (\hat{z}_0, w_0)(u = \pi + \tau/2). \quad (41)$$

I will decompose the mapping P into the following two parts

$$P_1 : (\hat{z}_0, w_0)(u = -\pi + \tau/2) \mapsto (\hat{\varrho}_0(u = \tau/2), \phi_0(u = \tau/2), \eta),$$

and

$$P_2 : (\hat{\varrho}_0(u = \tau/2), \phi_0(u = \tau/2), \eta) \mapsto (\hat{z}_0, w_0)(u = \pi + \tau/2),$$

so that $P = P_2 \circ P_1$. Here I have adopted a similar notation to the one used in (41). I further decompose P_1 into three parts setting $P_1 = P_i \circ P_{\text{cr}}^{\text{ext}} \circ P_o$ where $P_{\text{cr}}^{\text{ext}}$ is as in Definition 1 and where

- P_o is the “outer” map

$$P_o : (\hat{z}_0, w_0)(u = -\pi + \tau/2) \mapsto (\hat{z}_0, w_0)(\hat{u} = -\hat{u}_*);$$

- P_i is the “inner” map:

$$P_i : (\hat{\varrho}_0, \phi_0)(\hat{u} = \hat{u}_*) \mapsto (\hat{\varrho}_0, \phi_0)(u = \tau/2).$$

In principle, to make sense of $P_i \circ P_{\text{cr}}^{\text{ext}}$, I should include η in the argument of P_i , but by symmetry P_i applies as two identical copies on the components $\eta = \pm 1$. For ease of notation I just think of P_i acting on the P_{cr} -part only (recall the definition of P_{cr} in Definition 1). The maps are illustrated in Fig. 2.

By assumption (A3) $f(u) = f(\tau - u)$ and so the equations of motion are invariant with respect to the following transformation

$$\begin{aligned} x &\mapsto x, \\ y &\mapsto -y, \\ u &\mapsto \tau - u. \end{aligned}$$

This is the time-reversible \mathcal{T}_τ -symmetry (3) viewed as an action on $(x, y) = (x, y)(u)$. From this follows:

PROOF. By choice of \hat{u}_* , the function $\hat{\Omega}(\hat{u})^{-2}$ is bounded from above by some constant $c = c(\hat{u}_*)$. Therefore

$$\frac{\hat{\Omega}(\hat{u})^{-2p}}{\hat{\Omega}(\hat{u})^{-2q}} = \hat{\Omega}(\hat{u})^{-2(p-q)} \leq c^{p-q},$$

given that $p > q$.

Lemma 6. *Let $q \in \overline{\mathbb{R}}_+$. Given an integrable function $r = r(\hat{u})$, $\hat{u} \in [\hat{u}_*, \mu^{-2}\tau/2]$ satisfying the following estimate*

$$|r(\hat{u})| \leq \hat{\Omega}(\hat{u})^{-2q}, \quad \hat{u} \in [\hat{u}_*, \mu^{-2}\tau/2].$$

If $q < 1$ then there exists a $c_1 = c_1(q)$ so that

$$\left| \int_{\hat{u}_*}^{\epsilon^{-2/3}\delta\tau/2} r(\hat{u})d\hat{u} \right| \leq c_1\mu^{-2(1-q)}.$$

If $q = 1$ then there exists a c_2 so that

$$\left| \int_{\hat{u}_*}^{\epsilon^{-2/3}\delta\tau/2} r(\hat{u})d\hat{u} \right| \leq c_2 \ln(\mu^{-2}\hat{u}_*^{-1}). \quad (43)$$

Finally if $q > 1$ then the corresponding integral is uniformly bounded with respect to ϵ : There exists a c_3 so that

$$\left| \int_{\hat{u}_*}^{\epsilon^{-2/3}\delta\tau/2} r(\hat{u})d\hat{u} \right| \leq c_3(q-1)^{-1}\hat{u}_*^{1-q}.$$

PROOF. For $q \neq 1$ I have that

$$\begin{aligned} \left| \int_{\hat{u}_*}^{\mu^{-2}\tau/2} r(\hat{u})d\hat{u} \right| &\leq \int_{\hat{u}_*}^{\mu^{-2}\tau/2} |r(\hat{u})|d\hat{u} \\ &\leq \int_{\hat{u}_*}^{\mu^{-2}\tau/2} \hat{\Omega}(\hat{u})^{-2q}d\hat{u} \\ &= (\text{Use (21)}) \\ &\leq c_1 (\mu^2)^{q-1} \int_{u_*}^{\tau/2} \vartheta(u)^{-q}du, \end{aligned}$$

setting $u_* = \mu^2 \hat{u}_*$. Here c_1 is some constant depending only q . I then use that $u\vartheta(u)^{-1} = 1 + \mathcal{O}(u)$ and therefore $\vartheta(u)^{-1} = u^{-1} + \mathcal{O}(1)$ cf. (22) for $u > 0$ to conclude

$$\left| \int_{\hat{u}_*}^{\mu^{-2}\tau/2} r(\hat{u}) d\hat{u} \right| \leq c_2 c_1 \mu^{2(q-1)} (1-q)^{-1} \left((\tau/2)^{1-q} - u_*^{1-q} \right), \quad (44)$$

for some constant c_2 again depending only on q . If $q < 1$ then $(\tau/2)^{1-q}$ dominates the last factor and I obtain the first result of the lemma:

$$\left| \int_{\hat{u}_*}^{\mu^{-2}\tau/2} r(\hat{u}) d\hat{u} \right| \leq 2c_2 c_1 \mu^{-2(q-1)} (1-q)^{-1} (\tau/2)^{1-q},$$

\hat{u}_* sufficiently small. For $q > 1$ the last term in (44) dominates for ϵ small and so upon inserting $u_* = \mu^2 \hat{u}_*$ I obtain

$$\left| \int_{\hat{u}_*}^{\mu^{-2}\tau/2} r(\hat{u}) d\hat{u} \right| \leq 2c_2 c_1 (q-1)^{-1} \hat{u}_*^{1-q},$$

completing the third part for \hat{u}_* sufficiently small. The case $q = 1$ is also the consequence of a simple calculation.

Recall the form of $\hat{\Lambda}$ in (17):

$$\begin{aligned} \hat{\Lambda} &= \zeta_0(\hat{u}, \hat{v}_0, \hat{\varrho}_0) + \rho_0(\hat{u}, \hat{\varrho}_0, \phi_0) + \mathcal{O}(\hat{\Omega}^{-7/2} \delta^{9/4}), \\ \rho_0 &= \delta^{3/4} \hat{\Omega}(\hat{u})^{-1/2} \rho_{01} + \delta^{3/2} \hat{\Omega}(\hat{u})^{-2} \rho_{02}, \end{aligned} \quad (45)$$

where I have already used Lemma 5 to say that $\hat{\Omega}^{-5} \leq c^{3/4} \hat{\Omega}^{-7/2}$. The averages of ρ_{01} and ρ_{02} in (19) and (20) are easily computed:

$$\begin{aligned} \bar{\rho}_{01}(u, \hat{\varrho}_0) &= \frac{1}{2\pi} \int_0^{2\pi} \rho_{01}(u, \hat{\varrho}_0, \tau) d\tau = 0, \\ \bar{\rho}_{02}(u, \hat{\varrho}_0) &= \frac{1}{2\pi} \int_0^{2\pi} \rho_{02}(u, \hat{\varrho}_0, \tau) d\tau = \frac{3}{4} (1 + \mathcal{O}(u)) \hat{\varrho}_0^2 + \frac{3}{2} u^3 M_{02}(u) (1 + \mathcal{O}(u)) \hat{\varrho}_0^2 \\ &\quad + \frac{1}{2} u M_0(u) (1 + \mathcal{O}(u)) \hat{\varrho}_0^2. \end{aligned}$$

Therefore

$$\begin{aligned}\bar{\rho}_0(\hat{u}, \hat{\varrho}_0) &= \delta^{3/2} \tilde{\Omega}_0(\hat{u})^{-2} \left(\frac{3}{4}(1 + \mathcal{O}(u)) + \frac{3}{2}u^3 M_{02}(u)(1 + \mathcal{O}(u)) \right. \\ &\quad \left. + \frac{1}{2}u M_0(u)(1 + \mathcal{O}(u)) \right) \hat{\varrho}_0^2 \\ &= \frac{3}{4} \delta^{3/2} \hat{\Omega}(\hat{u})^{-2} (1 + \mathcal{O}(u)) \hat{\varrho}_0^2.\end{aligned}$$

I also set $\tilde{\rho}_0 = \rho_0 - \bar{\rho}_0$ so that $\tilde{\rho}_0$ has zero average. I first realise the following:

Lemma 7. *The order of $\tilde{\rho}_0$ is $\hat{\Omega}^{-1/2} \delta^{3/4}$. Similarly the order of $\partial_{\hat{u}} \tilde{\rho}_0$ is $\hat{\Omega}^{-5/2} \delta^{3/4}$.*

PROOF. The term with ρ_{01} dominates the expression for ρ_0 cf. $\delta^{3/2} \ll \delta^{3/4}$ and using (42) to say that $\hat{\Omega}^{-3/2} \leq c^{1/2} \hat{\Omega}^{-1/2}$. This gives the first claim. The next claim follows from similar arguments upon differentiation with respect to \hat{u} :

$$\begin{aligned}\partial_{\hat{u}} \tilde{\rho} &= -\frac{1}{2} \delta^{3/4} \hat{\Omega}(\hat{u})^{-5/2} (1 + \mathcal{O}(u)) \rho_{01}(u, \hat{\varrho}_0, \phi_0) + \delta^{3/4} \mu^2 \hat{\Omega}(\hat{u})^{-1/2} \partial_u \rho_{01}(u, \hat{\varrho}_0, \phi_0) \\ &\quad - 2\delta^{3/2} \hat{\Omega}(\hat{u})^{-4} (1 + \mathcal{O}(u)) \rho_{02}(u, \hat{\varrho}_0, \phi_0) + \delta^{3/2} \mu^2 \hat{\Omega}(\hat{u})^{-2} \partial_u \rho_{02}(u, \hat{\varrho}_0, \phi_0)\end{aligned}$$

Here I have used (23). I complete the result by replacing μ^2 by (24) to obtain

$$\partial_{\hat{u}} \tilde{\rho} = \mathcal{O}(\hat{\Omega}^{-5/2} \delta^{3/4} + \hat{\Omega}^{-4} \delta^{3/2}) = \mathcal{O}(\hat{\Omega}^{-5/2} \delta^{3/4}).$$

To push the phase-dependency to higher order, I then use the following generating function

$$G(\hat{u}, \hat{\nu}_1, \hat{\varrho}_0, \phi_1) = \delta^{-3/2} \hat{u} \hat{\nu}_1 + \hat{\varrho}_0 \phi_1 + \hat{\Omega}(\hat{u})^{-1} \int_0^{\phi_1} \tilde{\rho}_0(\hat{u}, \hat{\varrho}_0, \tau) d\tau,$$

to generate a transformation given as the solution to the equations:

$$\begin{aligned}\hat{u}_1 &= \delta^{3/2} \partial_{\hat{\nu}_1} G = \hat{u}, \\ \hat{\nu}_0 &= \delta^{3/2} \partial_{\hat{u}} G = \hat{\nu}_1 + \delta^{3/2} \hat{\Omega}(\hat{u})^{-1} \int_0^{\phi_1} \partial_{\hat{u}} \tilde{\rho}_0(\hat{u}, \hat{\varrho}_0, \tau) d\tau \\ &= \hat{\nu}_1 + \mathcal{O}(\hat{\Omega}(\hat{u})^{-7/2} \delta^{9/4}),\end{aligned}\tag{46}$$

using Lemma 7 in the last equality, and

$$\begin{aligned}\hat{\varrho}_1 &= \partial_{\phi_1} G = \hat{\varrho}_0 + \hat{\Omega}(\hat{u})^{-1} \tilde{\rho}_0(\hat{u}, \hat{\varrho}_0, \phi_1), \\ \phi_0 &= \partial_{\hat{\varrho}_0} G = \phi_1 + \hat{\Omega}(\hat{u})^{-1} \int_0^{\phi_1} \partial_{\hat{\varrho}_0} \tilde{\rho}_0(\hat{u}, \hat{\varrho}_0, \tau) d\tau.\end{aligned}$$

I obtain

$$\hat{\Lambda} = \zeta_1(\hat{u}, \hat{\nu}_1, \hat{\varrho}_1) + \rho_1(\hat{u}, \hat{\nu}_1, \hat{\varrho}_1) + \mathcal{O}(\hat{\Omega}^{-7/2} \delta^{9/4}),$$

where

$$\begin{aligned}\zeta_1(\hat{u}, \hat{\nu}_1, \hat{\varrho}_1) &= \hat{\nu}_1 + \hat{\Omega}(\hat{u}) \hat{\varrho}_1 + \bar{\rho}_0(\hat{u}, \hat{\varrho}_1) = \hat{\nu}_1 + \hat{\Omega}(\hat{u}) \hat{\varrho}_1 + \frac{3}{4} \delta^{3/2} \hat{\Omega}(\hat{u})^{-2} (1 + \mathcal{O}(u)) \hat{\varrho}_0^2, \\ \rho_1(\hat{u}, \hat{\varrho}_1, \phi_1) &= \bar{\rho}_0(\hat{u}, \hat{\varrho}_0) - \bar{\rho}_0(\hat{u}, \hat{\varrho}_1) + \tilde{\rho}_0(\hat{u}, \hat{\varrho}_0, \phi_0) - \tilde{\rho}_0(\hat{u}, \hat{\varrho}_0, \phi_1) \\ &\quad + \delta^{3/2} \hat{\Omega}(\hat{u})^{-1} \int_0^{\phi_1} \partial_{\hat{u}} \tilde{\rho}_0(\hat{u}, \hat{\varrho}_0, \tau) d\tau.\end{aligned}$$

In the following lemma I estimate ρ_1 .

Lemma 8. *The remainder $\rho_1 = \rho_1(\hat{u}, \hat{\varrho}_1, \phi_1)$ takes the following form*

$$\rho_1(\hat{u}, \hat{\varrho}_1, \phi_1) = \delta^{3/2} \hat{\Omega}(\hat{u})^{-2} \partial_{\phi_0} \rho_{01}(u, \hat{\varrho}_1, \tau) \int_0^{\phi_1} \partial_{\hat{\varrho}_0} \rho_{01}(u, \hat{\varrho}_1, \tau) d\tau + \mathcal{O}(\hat{\Omega}^{-7/2} \delta^{9/4}).$$

PROOF. Firstly,

$$\begin{aligned}\bar{\rho}_0(\hat{u}, \hat{\varrho}_0) - \bar{\rho}_0(\hat{u}, \hat{\varrho}_1) &= \frac{3}{4} \delta^{3/2} \hat{\Omega}(\hat{u})^{-2} (1 + \mathcal{O}(u)) (\hat{\varrho}_0 + \hat{\varrho}_1) (\hat{\varrho}_0 - \hat{\varrho}_1) \\ &= -\frac{3}{4} \delta^{3/2} \hat{\Omega}(\hat{u})^{-3} (1 + \mathcal{O}(u)) (\hat{\varrho}_0 + \hat{\varrho}_1) \tilde{\rho}(\hat{u}, \hat{\varrho}_0, \phi_1) \\ &= \mathcal{O}(\hat{\Omega}^{-7/2} \delta^{9/4}),\end{aligned}$$

using Lemma 7. Next using Taylor's theorem:

$$\begin{aligned}\tilde{\rho}_0(\hat{u}, \hat{\varrho}_0, \phi_0) - \tilde{\rho}_0(\hat{u}, \hat{\varrho}_0, \phi_1) &= \partial_{\phi_0} \tilde{\rho}_0(\hat{u}, \hat{\varrho}_0, \phi_1) (\phi_0 - \phi_1) + \int_0^1 (1-s) \\ &\quad \times \partial_{\phi_0}^2 \tilde{\rho}_0(\hat{u}, \hat{\varrho}_0, \phi_1 + s(\phi_0 - \phi_1)) (\phi_0 - \phi_1)^2 ds \\ &= \hat{\Omega}(\hat{u})^{-1} \partial_{\phi_0} \tilde{\rho}_0(\hat{u}, \hat{\varrho}_0, \phi_1) \int_0^{\phi_1} \partial_{\hat{\varrho}_0} \tilde{\rho}_0(\hat{u}, \hat{\varrho}_0, \tau) d\tau + \mathcal{O}(\hat{\Omega}(\hat{u})^{-7/2} \delta^{9/4}) \\ &= \hat{\Omega}(\hat{u})^{-1} \partial_{\phi_0} \tilde{\rho}_0(\hat{u}, \hat{\varrho}_1, \phi_1) \int_0^{\phi_1} \partial_{\hat{\varrho}_0} \tilde{\rho}_0(\hat{u}, \hat{\varrho}_1, \tau) d\tau + \mathcal{O}(\hat{\Omega}(\hat{u})^{-7/2} \delta^{9/4}),\end{aligned}$$

having here also used that $\hat{\varrho}_0 - \hat{\varrho}_1, \phi_0 - \phi_1 = \mathcal{O}(\hat{\Omega}(\hat{u})^{-3/2}\delta^{3/4})$ cf. Lemma 7. I complete the result by using (46) and noticing the following:

$$\begin{aligned} \hat{\Omega}(\hat{u})^{-1} \partial_{\phi_0} \tilde{\rho}_0(\hat{u}, \hat{\varrho}_1, \phi_1) \int_0^{\phi_1} \partial_{\hat{\varrho}_0} \tilde{\rho}_0(\hat{u}, \hat{\varrho}_1, \tau) d\tau &= \delta^{3/2} \hat{\Omega}(\hat{u})^{-2} \partial_{\phi_0} \rho_{01}(u, \hat{\varrho}_1, \tau) \\ &\times \int_0^{\phi_1} \partial_{\hat{\varrho}_0} \rho_{01}(u, \hat{\varrho}_1, \tau) d\tau + \mathcal{O}(\hat{\Omega}^{-7/2} \delta^{9/4}), \end{aligned}$$

which follows from (45).

One more averaging step is needed to push the order of the error below $\hat{\Omega}^{-2} \delta^{3/2}$: Note that its contribution matter cf. (43) on the time scale considered for P_i . I therefore define

$$\rho_{12}(u, \hat{\varrho}_1, \phi_1) = \partial_{\phi_0} \rho_{01}(u, \hat{\varrho}_1, \phi_1) \int_0^{\phi_1} \partial_{\hat{\varrho}_0} \rho_{01}(u, \hat{\varrho}_1, \tau) d\tau,$$

so that

$$\rho_1(\hat{u}, \hat{\varrho}_1, \phi_1) = \delta^{3/2} \hat{\Omega}(\hat{u})^{-2} \rho_{12}(u, \hat{\varrho}_1, \phi_1) + \mathcal{O}(\hat{\Omega}^{-7/2} \delta^{9/4}).$$

The average of ρ_{12} is easily computed given (19):

$$\begin{aligned} \bar{\rho}_{12}(u, \hat{\varrho}_1) &= -\frac{15}{4}(1 + \mathcal{O}(u))\hat{\varrho}_1^2 - \frac{1}{8}(1 + \mathcal{O}(u)) - \frac{3}{4}u^2 M_0(u)^2(1 + \mathcal{O}(u))\hat{\varrho}_1^2 \\ &\quad - \frac{3}{2}u M_0(1 + \mathcal{O}(u))\hat{\varrho}_1 \\ &= -\frac{15}{4}(1 + \mathcal{O}(u))\hat{\varrho}_1^2 - \frac{1}{8}(1 + \mathcal{O}(u)). \end{aligned}$$

I set $\tilde{\rho}_1 = \rho_1 - \bar{\rho}_1$ and use the following generating function

$$G(\hat{u}_1, \hat{\nu}, \hat{\varrho}_1, \phi) = \delta^{-3/2} \hat{u}_1 \hat{\nu} + \hat{\varrho}_1 \phi + \hat{\Omega}(\hat{u}_1)^{-1} \int_0^{\phi} \tilde{\rho}_1(\hat{u}_1, \hat{\varrho}_1, \tau) d\tau,$$

to generate a final transformation $(\hat{u}_1, \hat{\nu}_1, \hat{\varrho}_1, \phi_1) \mapsto (\hat{u}, \hat{\nu}, \hat{\varrho}, \phi)$. This gives the following form of the Hamiltonian in the new variables:

$$\hat{\Lambda} = \zeta(\hat{u}, \hat{\nu}, \hat{\varrho}) + \mathcal{O}(\hat{\Omega}^{-7/2} \delta^{9/4}),$$

where

$$\zeta(\hat{u}, \hat{\nu}, \hat{\varrho}) = h_1(\hat{u}, \hat{\nu}, \hat{\varrho}) + \bar{\rho}_1(\hat{u}, \hat{\varrho}) = \hat{\nu} + \hat{\Omega}(\hat{u})\hat{\varrho} - 3\delta^{3/2} \hat{\Omega}(\hat{u})^{-2} (1 + \mathcal{O}(u)) \hat{\varrho}^2.$$

I have here ignored the term $-\frac{1}{8}\delta^{3/2}\hat{\Omega}(\hat{u})^{-2}(1 + \mathcal{O}(u))$ that only depends on u ; it can be removed by a further translation of $\hat{\nu}$. The equations of motion are

$$\frac{d\hat{\varrho}}{d\hat{u}} = \mathcal{O}(\hat{\Omega}^{-7/2}\delta^{3/4}), \quad (47)$$

$$\frac{d\phi}{d\hat{u}} = -\delta^{-3/2}\hat{\Omega}(\hat{u}) + 6\hat{\Omega}(\hat{u})^{-2}(1 + \mathcal{O}(u))\hat{\varrho} + \mathcal{O}(\hat{\Omega}^{-7/2}\delta^{3/4}). \quad (48)$$

The mapping P_i describes the assignment $(\hat{\varrho}_0, \phi_0)(\hat{u} = \hat{u}_*) \mapsto (\hat{\varrho}_0, \phi_0)(\hat{u} = \mu^{-2}\tau/2)$. To approximate this I will solve the truncation of the transformed differential equations (47) and (48) from $\hat{u} = \hat{u}_*$ to $\hat{u} = \mu^{-2}\tau/2$. This is adequate because of the following:

Lemma 9. *$\hat{\varrho}_0$ is conserved on the interval from $\hat{u} = \hat{u}_*$ to $\hat{u} = \mu^{-2}\tau/2$ with accuracy of $\delta^{3/4}$.*

PROOF. I use Lemma 6 with $|r(\hat{u})| \leq \hat{\Omega}^{-7/2}$ ($q = 7/4 > 1$) together with (47) to conclude that the variation $\Delta\hat{\varrho}$ of $\hat{\varrho}$ on the given interval can be estimated from above by

$$|\Delta\hat{\varrho}| \leq c\delta^{3/4}.$$

Since $\hat{\varrho} - \hat{\varrho}_0 = \mathcal{O}(\delta^{3/4})$ this completes the result.

By a similar argument, I estimate the effect of the remainder in (48) by $\mathcal{O}(\delta^{3/4})$ and I compute the variation in ϕ by

$$\begin{aligned} \phi(\epsilon^{-2/3}\delta\tau/2) &= \phi(\hat{u}_*) - \int_{\hat{u}_*}^{\mu^{-2}\tau/2} \left(\delta^{-3/2}\hat{\Omega}(\hat{u}) - 6\hat{\Omega}(\hat{u})^{-2}(1 + \mathcal{O}(u)) \right) \hat{\varrho}_0 \\ &\quad + \int_{\hat{u}_*}^{\mu^{-2}\tau/2} \mathcal{O}(\hat{\Omega}(\hat{u})^{-2}\delta^{3/4}) d\hat{u} + \mathcal{O}(\delta^{3/4}). \end{aligned} \quad (49)$$

The remainder $\mathcal{O}(\hat{\Omega}(\hat{u})^{-2}\delta^{3/4})$ in the integral comes from $\hat{\varrho}(\hat{u}) = \hat{\varrho}_0 + \mathcal{O}(\delta^{3/4})$ with $\hat{\varrho}_0 = \text{const.}$ on this interval. This term can be estimated from above by a term of order $\delta^{3/4} \ln \epsilon^{-1}$ cf. Lemma 6 with $q = 1$. The following lemma gives asymptotics of the two other integrals appearing in (49).

Lemma 10.

$$\delta^{-3/2} \int_{\hat{u}_*}^{\mu^{-2}\tau/2} \hat{\Omega}(\hat{u}) d\hat{u} = \sqrt{2}\epsilon^{-1}e_1 - \frac{2\sqrt{2}}{3}\delta^{-3/2}\hat{u}_*^{3/2} + \mathcal{O}(\epsilon^{2/3}\delta^{-5/2}).$$

with

$$e_1 = \int_0^{\tau/2} \vartheta(u)^{1/2} du,$$

with $\vartheta = \vartheta(u) = u + \mathcal{O}(u^2)$ (22). Moreover, there exists a positive constant e_2 independent of ϵ such that

$$\int_{\hat{u}_*}^{\mu^{-2}\tau/2} \hat{\Omega}(\hat{u})^{-2}(1 + \mathcal{O}(u)) d\hat{u} = \frac{1}{3} \ln(\epsilon^{-1}) - \frac{1}{2} \ln(\delta^{-1}\hat{u}_*) + \mathcal{O}(\mu^2).$$

PROOF. I use (22) to write $\hat{\Omega}(\hat{u})$ as $\sqrt{2}\mu^{-1}\vartheta(u)^{1/2}$ with $\vartheta(0) = 0$, $\vartheta'(0) = 1$. Therefore

$$\begin{aligned} \delta^{-3/2} \int_{\hat{u}_*}^{\mu^{-2}\tau/2} \hat{\Omega}(\hat{u}) d\hat{u} &= \sqrt{2}\epsilon^{-1} \int_{u_*}^{\tau/2} \vartheta(u)^{1/2} du \\ &= \sqrt{2}\epsilon^{-1}e_1 - \sqrt{2}\epsilon^{-1} \int_0^1 \vartheta(u_*s)^{1/2} ds u_*, \end{aligned} \quad (50)$$

here $u_* = \mu^2\hat{u}_*$. For the last integral I use the following

$$\int_0^1 u_*^{-1/2} \vartheta(u_*s)^{1/2} ds u_*^{3/2} = \int_0^1 (1 + \mathcal{O}(u_*^{3/2}s^{3/2})) ds u_*^{3/2} = \frac{2}{3}u_*^{3/2} + \mathcal{O}(u_*^{5/2}). \quad (51)$$

Inserting this back into (50) completes the first part of the proof.

For the second part the integral is written as

$$\int_{\hat{u}_*}^{\mu^{-2}\tau/2} \hat{\Omega}(\hat{u})^{-2}(1 + \mathcal{O}(u)) d\hat{u} = \frac{1}{2} \int_{u_*}^{\tau/2} \vartheta(u)^{-1}(1 + \mathcal{O}(u)) du.$$

I write $\vartheta(u)^{-1} = u^{-1} + \mathcal{O}(1)$ for u small so that

$$\begin{aligned} \int_{\hat{u}_*}^{\epsilon^{-2/3}\delta\tau/2} \hat{\Omega}(\hat{u})^{-2}(1 + \mathcal{O}(u)) d\hat{u} &= \frac{1}{2} \int_{u_*}^{\tau/2} u^{-1} du + \mathcal{O}(1) \\ &= \frac{1}{2} \ln(\mu^{-2}\hat{u}_*^{-1}) + \mathcal{O}(1) \\ &= \frac{1}{3} \ln \epsilon^{-1} - \frac{1}{2} \ln(\delta^{-1}\hat{u}_*) + \mathcal{O}(1), \end{aligned}$$

using that $\mu^2 = \epsilon^{2/3}\delta^{-1}$. The $\mathcal{O}(1)$ -term is smooth as a function of u_* and can therefore be written as $\text{const.} + \mathcal{O}(u_*)$ for u_* small by Taylor's theorem. Writing the constant as $\frac{1}{3}\ln e_2$ completes the proof.

I can therefore write (49) as

$$\begin{aligned} \phi_0(\mu^{-2}\tau/2) &= \phi_0(\hat{u}_*) - \sqrt{2}\epsilon^{-1}e_1 + 2\ln(e_2\epsilon^{-1})\hat{\rho}_0 + \frac{2\sqrt{2}}{3}\delta^{-3/2}\hat{u}_*^{3/2} - 3\ln(\delta^{-1}\hat{u}_*)\hat{\rho}_0 \\ &\quad + \mathcal{O}(\delta^{3/4}\ln\epsilon^{-1}), \end{aligned}$$

using $\phi_0 = \phi + \mathcal{O}(\delta^{3/4})$. I collect the result about P_i in the following proposition:

Proposition 1. *Suppose that*

$$\delta^{3/4}\ln\epsilon^{-1} \ll 1, \quad (52)$$

then

$$\begin{aligned} P_i(\hat{\rho}_0, \phi_0) &= \left(\phi_0 - \sqrt{2}\epsilon^{-1}e_1 + (2\ln(e_2\epsilon^{-1}) - 3\ln(\delta^{-1}\hat{u}_*))\hat{\rho}_0 + \frac{2\sqrt{2}}{3}\delta^{-3/2}\hat{u}_*^{3/2} \right) \\ &\quad + \mathcal{O}(\delta^{3/4}\ln\epsilon^{-1}). \end{aligned} \quad (53)$$

Remark 9. Note that it is possible to realise (52) without violating condition (39).

3.2. The outer map P_o

To approximate the outer map I continue as above for the inner map and apply averaging to (14). The details are delayed to App. Appendix B and I simply state the result:

Proposition 2. *Suppose that*

$$\delta^{3/2}\ln\epsilon^{-1} \ll 1,$$

then

$$\begin{aligned} P_o(\hat{z}_0, w_0) &= \left(w_0 - \epsilon^{-1}e_3 - (\ln(e_4\epsilon^{-1}) - \frac{3}{2}\ln(\delta^{-1}\hat{u}_*))\hat{z}_0 + \frac{2}{3}\delta^{-3/2}\hat{u}_*^{3/2} \right) \\ &\quad + \mathcal{O}(\delta^{3/2}\ln\epsilon^{-1}). \end{aligned} \quad (54)$$

Here

$$e_3 = \int_0^{\pi-\tau/2} (-f(-u))^{1/2} du,$$

while e_4 is another positive constant defined in App. Appendix B.

3.3. Fix point Eq.

To obtain periodic orbits I solve the following fix point equation

$$\chi(\hat{z}_0, w_0) = P(\hat{z}_0, w_0), \quad P = P_2 \circ P_1, \quad (55)$$

“up to symmetry” taking either $\chi = Id$ or $\chi = \gamma$ with γ the translation from Lemma 2. A simple calculation gives the following result:

Lemma 11. *If $\chi = \gamma$ in (55) then (\hat{z}_0, w_0) is a periodic-2 point.*

PROOF. It follows from the fact that $\gamma^2 = Id$ and that P is equivariant with respect to the action of γ .

To avoid having to invert the crossing map P_c^{ext} I re-write (55) as

$$P_1 \begin{pmatrix} \hat{z}_0 \\ w_0 \end{pmatrix} = P_2^{-1} \circ \chi \begin{pmatrix} \hat{z}_0 \\ w_0 \end{pmatrix} = E \circ P_1 \circ E \circ \chi \begin{pmatrix} \hat{z}_0 \\ w_0 \end{pmatrix},$$

by inverting P_2 using Lemma 4. The mapping P_1 is then replaced by $P_i \circ P_c^{\text{ext}} \circ P_o$, and then by using the equivariance of P_o and E with respect to the γ action, one easily verifies, that this equation can be solved by solving

$$P_i \circ P_c \circ P_o \begin{pmatrix} \hat{z}_0 \\ w_0 \end{pmatrix} = E \circ P_i \circ P_c \circ P_o \circ E \begin{pmatrix} \hat{z}_0 \\ w_0 \end{pmatrix}, \quad (56)$$

taking $\chi = Id$ if

$$\eta \left(P_o \begin{pmatrix} \hat{z}_0 \\ w_0 \end{pmatrix} \right) = \eta \left(P_o \circ E \begin{pmatrix} \hat{z}_0 \\ w_0 \end{pmatrix} \right),$$

and taking, cf. Lemma 2, $\chi = \gamma$, if this last equality, on the other hand, does not hold.

Remark 10. A solution (\hat{z}_0, w_0) to (56) always defines a periodic orbit. Solutions with $\chi = Id$ correspond to fix points of P , that is periodic orbits of (2) with periods $T = 2\pi\epsilon^{-1}$, where $(|x(u)|, y(u))$ remains close to the singular solution $(|x_s(u)|, y_s)$ (6). The periodic-2 points, that appear when one has to take χ to be γ , correspond to periodic orbits of twice the period: $T = 4\pi\epsilon^{-1}$. Here $(|x(u)|, y(u))$ is still close to the singular solution $(|x_s(u)|, y_s)$ (6), but in this case the motion alternates between being close to $\kappa(u)$ to being close to $-\kappa(u)$: It is γ -symmetric. The latter property is a consequence of the fact that if w_0 is shifted by π then λ is also shifted by π , cf. (27) and (33). By definition this changes the sign of η and what route is taken. The symmetry properties of the periodic orbits are also the subject of Proposition 4 below.

4. Solving the fix point Eq. (56) - Proof of the main results

The left hand side of Eq. (56)

Setting $w_0(-\hat{u}_*)$ from (54) equal to the $w_0(-\hat{u}_*)$ in (27) it is realised that the phase $l = l_l$, using here the subscript l to indicate “left”, is given as

$$l_l = w_0 - \epsilon^{-1}e_3 - \ln(e_4\epsilon^{-1})\hat{z}_0 + \frac{\pi}{2}, \quad (57)$$

ignoring the $\mathcal{O}(\delta^{3/2} \ln \epsilon^{-1})$ -remainder. The image $P_c \circ P_o \left(\begin{smallmatrix} \hat{z}_0 \\ w_0 \end{smallmatrix} \right)$ therefore becomes

$$\begin{aligned} \hat{\rho}_0 &= \frac{1}{2\pi} \ln \frac{1 + |p_l|^2}{2|\operatorname{Im} p_l|}, \\ \phi_0 &= -\frac{2\sqrt{2}}{3} \delta^{-3/2} \hat{u}_*^{3/2} + 3 \ln(\delta^{-1} \hat{u}_*) \hat{\rho}_0 - \theta_l, \end{aligned} \quad (58)$$

where $\theta_l = \theta(\hat{\rho}_0, p_l)$ (30), $p_l = p(\hat{z}_0, \lambda_l)$ (32) and

$$\lambda_l = 3\hat{z}_0 \ln 2 - \frac{\pi}{4} - \arg \Gamma(i\hat{z}_0) - l_l.$$

Given (57) it follows that λ_l can be written as

$$\lambda_l = -w_0 + \ln(e_4\epsilon^{-1})\hat{z}_0 + G(\hat{z}_0), \quad (59)$$

$$G(\hat{z}_0) = 3\hat{z}_0 \ln 2 - \frac{3\pi}{4} - \arg \Gamma(i\hat{z}_0) + \epsilon^{-1}e_3. \quad (60)$$

Finally applying P_i (53) to the Eqs. (58) gives the left hand side of (56) which is denoted by $(\hat{\rho}_l, \phi_l)$:

$$\hat{\rho}_l = \frac{1}{2\pi} \ln \frac{1 + |p_l|^2}{2|\operatorname{Im} p_l|}, \quad (61)$$

$$\phi_l = -\sqrt{2}\epsilon^{-1}e_1 + 2 \ln(e_2\epsilon^{-1})\hat{\rho}_l - \theta_l. \quad (62)$$

The right hand side of Eq. (56)

First the image of $P_o \circ E, (\hat{z}_0, w_0)(\mu^{-2}\tau + \hat{u}_*) = P_o \circ E(\hat{z}_0, w_0)$, is computed using (54):

$$\hat{z}_0(\mu^{-2}\tau + \hat{u}_*) = \hat{z}_0, \quad (63)$$

$$w_0(\mu^{-2}\tau + \hat{u}_*) = -w_0 - \epsilon^{-1}e_3 - \ln(e_4\epsilon^{-1})\hat{z}_0 + \frac{2}{3}\delta^{-3/2}\hat{u}_*^{3/2} + \frac{3}{2}\ln(\delta^{-1}\hat{u}_*),$$

having again ignored the $\mathcal{O}(\delta^{3/2} \ln \epsilon^{-1})$ -remainder. The phase $w_0(\mu^{-2}\tau + \hat{u}_*)$ is then, much as above, set equal to the phase w_0 in (27). This gives the following expression for the phase $l = l_r$, now using the subscript r to indicate “right”:

$$l_r = -w_0 - \epsilon^{-1}e_3 - \ln(e_4\epsilon^{-1})\hat{z}_0 + \frac{\pi}{2}. \quad (64)$$

Then upon applying P_c to (63) I obtain

$$\begin{aligned} \hat{\varrho}_0 &= \frac{1}{2\pi} \ln \frac{1 + |p_r|^2}{2|\operatorname{Im} p_r|}, \\ \phi_0 &= -\frac{2\sqrt{2}}{3} \delta^{-3/2} \hat{u}_*^{3/2} + 3 \ln(\delta^{-1} \hat{u}_*) \hat{\varrho}_0 - \theta_r, \end{aligned}$$

with $\theta_r = \theta(\hat{\varrho}_0, p_r)$ (30), $p_r = p(\hat{z}_0, \lambda_r)$ (32) and

$$\lambda_r = 3\hat{z}_0 \ln 2 - \frac{\pi}{4} - \arg \Gamma(i\hat{z}_0) - l_r.$$

Given (64) it follows that λ_r takes the following form:

$$\lambda_r = w_0 + \ln(e_4\epsilon^{-1})\hat{z}_0 + G(\hat{z}_0), \quad (65)$$

with G as in (60). Finally, $E \circ P_i$ is applied to this, using the approximation (53) for P_i . This finally gives the right hand side of (56) which is denoted by $(\hat{\varrho}_r, \phi_r)$

$$\hat{\varrho}_r = \frac{1}{2\pi} \ln \frac{1 + |p_r|^2}{2|\operatorname{Im} p_r|}, \quad (66)$$

$$\phi_r = \sqrt{2}\epsilon^{-1}e_1 - 2 \ln(e_2\epsilon^{-1})\hat{\varrho}_r + \theta_r. \quad (67)$$

Solving $\hat{\varrho}_l = \hat{\varrho}_r$ using (57) and (64)

The absolute value of p depends, cf. (32), only on \hat{z}_0 . Hence $|p_l| = |p_r|$. Setting $\hat{\varrho}_l = \hat{\varrho}_r = \hat{\varrho}_0$ I therefore conclude that

$$\sin \lambda_l = \pm \sin \lambda_r, \quad (68)$$

giving

$$\lambda_r = \begin{cases} \pm \lambda_l \\ \pi \pm \lambda_l \end{cases}. \quad (69)$$

Before continuing with solving the equation:

$$\phi_l = \phi_r \pmod{2\pi},$$

I first compute the trace of the Jacobian matrix $\partial P = \partial_{(\hat{z}_0, w_0)} P$ of the truncation of the mapping. For this Eq. (69) will be used. I compute this trace for two reasons: (i) To decide when the implicit function theorem can be applied to conclude that the solutions can be continued into true solutions of the non-truncated equations. (ii): To investigate the stability of the periodic orbits.

The Jacobian of P

The function η is locally constant so it can be ignored completely in the calculations.

Lemma 12. *Suppose $1 + p^2 \neq 0$. If (i) $\lambda_r = \lambda_l$ or $\pi + \lambda_l$ then*

$$\begin{aligned} \text{tr}(\partial P) = 2 + 4B & \left((2 \ln(e_2 \epsilon^{-1}) - q)(\ln(e_4 \epsilon^{-1}) + g)B \right. \\ & \left. + A(2 \ln(e_2 \epsilon^{-1}) - q) + D(\ln(e_4 \epsilon^{-1}) + g) + C \right). \end{aligned} \quad (70)$$

Here

$$g = g(\hat{z}_0) = \partial_{\hat{z}_0} G = 3 \ln 2 - \partial_{\hat{z}_0} \arg \Gamma(i \hat{z}_0), \quad (71)$$

and

$$q = q(\hat{\varrho}_0) = \partial_{\hat{\varrho}_0} Q = 7 \ln 2 - \partial_{\hat{\varrho}_0} \arg \Gamma(2i \hat{\varrho}_0), \quad (72)$$

are the derivatives of $G = G(\hat{z}_0)$ (60) and $Q = Q(\hat{\varrho}_0)$ (31) with respect to \hat{z}_0 respectively $\hat{\varrho}_0$. Moreover

$$A = \partial_{\hat{z}_0} \varrho_0(\hat{z}_0, \lambda) = \frac{|p|^2 + 1}{2|p|^2}, \quad (73)$$

$$B = \partial_{\lambda} \varrho_0(\hat{z}_0, \lambda) = -\frac{1}{2\pi} \cot \lambda, \quad (74)$$

$$C = \partial_{\hat{z}_0} (\arg(1 + p^2)) = \frac{2\pi(1 + |p|^2) \sin(2\lambda)}{|1 + p^2|^2}, \quad (75)$$

$$D = \partial_{\lambda} (\arg(1 + p^2)) = \frac{2|p|^2 (\cos(2\lambda) + |p|^2)}{|1 + p^2|^2}. \quad (76)$$

These functions are all evaluated for $p = p_l(\hat{z}_0, \lambda)$ and $\lambda = \lambda_l(\hat{z}_0, w_0)$. On the other hand, if (ii) $\lambda_r = -\lambda_l$ or $\pi - \lambda_l$ then

$$\text{tr}(\partial P) = 2 - 4(2 \ln(e_2 \epsilon^{-1}) - g)(\ln(e_4 \epsilon^{-1}) + q)B^2. \quad (77)$$

Remark 11. Note that the lemma, cf. (69), considers all of the four different scenarios.

PROOF. I use the expressions for $(\hat{\rho}_l, \phi_l)$ in (61) and (62) and $(\hat{\rho}_r, \phi_r)$ in (66) and (67). For case (ii) I also use that the functions A and D are even with respect to λ . The functions B and C are, on the other hand, odd.

Eqs. (70) and (77) are asymptotically of the form:

$$\text{tr}(\partial P) - 2 = \mp 8 \ln^2(\epsilon^{-1})B^2 + \mathcal{O}(\ln \epsilon^{-1}), \quad (78)$$

Therefore

Lemma 13. Consider all solutions of (55) with λ_l belonging to $[c^{-1}, \pi/2 - c^{-1}] \cup [\pi/2 + c^{-1}, \pi - c^{-1}]$ or a π -translation of this set. These periodic orbits are all unstable for ϵ sufficiently small.

PROOF. A sufficient condition for instability is that

$$|\text{tr} \partial P| > 2, \quad (79)$$

using also here that $\text{tr}(\partial P)^2 = \text{tr}^2 \partial P - 2$. For the considered λ_l -values $|b| \geq c_1^{-1} > 0$, c_1 independent of ϵ , and so (79) can always be archived by taking ϵ sufficiently small cf. (78).

Stable orbits can therefore only occur if λ_l is near $\pi/2$ (or $3\pi/2$ by symmetry) where b (74) is small. I consider this in Section 4.3.

4.1. Unstable solutions - Part 1° of the main result

In this section I will find unstable periodic orbits. I continue from (69) and start by dividing the presentation into two separate cases:

- Case (i) where $\lambda_r = \lambda_l$ or $\lambda_r = \pi + \lambda_l$;
- case (ii) where $\lambda_r = -\lambda_l$ or $\lambda_r = \pi - \lambda_l$.

These two cases cover all the possible cases. They also correspond to solutions with different symmetry properties (see Section 4.2 below).

Consider first case (i) and $\lambda_r = \lambda_l$. Then by using (59) and (65)

$$w_0 = 0 \quad \text{or} \quad \pi.$$

If $\lambda_r = \pi + \lambda_l$ then

$$w_0 = \pi/2 \quad \text{or} \quad 3\pi/2.$$

It follows that if

$$w_0 = 0 \quad \text{or} \quad \pi/2 \quad \text{mod } \pi,$$

then $\lambda_r = \lambda_l \text{ mod } \pi$. Also

$$p_r = \pm p_l. \tag{80}$$

On the other hand, for case (ii) with $\lambda_r = -\lambda_l$ then

$$2 \ln(e_4 \epsilon^{-1}) \hat{z}_0 + 2G(\hat{z}_0) = 0 \quad \text{mod } 2\pi,$$

using (59) and (65). For $\lambda_r = \pi - \lambda_l$ then I similarly get

$$2 \ln(e_4 \epsilon^{-1}) \hat{z}_0 + 2G(\hat{z}_0) = \pi \quad \text{mod } 2\pi.$$

It therefore follows that solutions to the equation

$$2 \ln(e_4 \epsilon^{-1}) \hat{z}_0 + 2G(\hat{z}_0) = 0 \quad \text{mod } \pi,$$

solve $\lambda_r = -\lambda_l \text{ mod } \pi$. Also

$$p_r = \pm \bar{p}_l. \tag{81}$$

Next, I consider the equation $\phi_l = \phi_r$ and use (62) and (67). In case (i) I use (80) so that $\theta_r = \theta_l$, $\theta = \theta(\hat{\varrho}_0, p)$ (30) depending only on p^2 , and conclude that

$$-\sqrt{2} \epsilon^{-1} e_1 + 2 \ln(e_2 \epsilon^{-1}) \hat{\varrho}_0 - \theta_l = 0 \quad \text{mod } \pi,$$

with $\theta_l = \theta(\hat{\varrho}_0, p_l)$, $p_l = (e^{2\pi \hat{z}_0} - 1)^{1/2} e^{i\lambda_l}$.

For (ii) I similarly use (81) so that $\theta_r = Q - \arg(1 + p_r^2) = Q + \arg(1 + p_l^2)$ and therefore

$$-\sqrt{2} \epsilon^{-1} e_1 + 2 \ln(e_2 \epsilon^{-1}) \hat{\varrho}_0 - Q = 0 \quad \text{mod } \pi,$$

with $Q = Q(\hat{\varrho}_0)$ (31). I collect the results in the following proposition:

Proposition 3. *Periodic orbits can be obtained as*

(i)

$$w_0 = 0, \pi/2 \bmod \pi, \quad (82)$$

$$F_i^{(2)}(\hat{z}_0) = 0 \bmod \pi, \quad (83)$$

where

$$F_i^{(2)}(\hat{z}_0) \equiv -\sqrt{2}\epsilon^{-1}e_1 + 2\ln(e_2\epsilon^{-1})\hat{\varrho}_0(\hat{z}_0, \lambda_l(\hat{z}_0)) - \theta(\hat{z}_0, p(\hat{z}_0, \lambda_l(\hat{z}_0))). \quad (84)$$

(ii)

$$F_{ii}^{(1)}(\hat{z}_0) = 0 \bmod \pi, \quad (85)$$

$$F_{ii}^{(2)}(\hat{z}_0, \lambda_l) = 0 \bmod \pi, \quad (86)$$

where

$$\begin{aligned} F_{ii}^{(1)}(\hat{z}_0) &\equiv 2\ln(e_4\epsilon^{-1})\hat{z}_0 + 2G(\hat{z}_0), \\ F_{ii}^{(2)}(\hat{z}_0, \lambda) &\equiv -\sqrt{2}\epsilon^{-1}e_1 + 2\ln(e_2\epsilon^{-1})\hat{\varrho}_0 - Q(\hat{z}_0). \end{aligned}$$

The action ϱ_0 is given by (28):

$$\varrho_0 = \frac{1}{2\pi} \ln \frac{1 + |p_l|^2}{2|\operatorname{Im} p_l|} = \frac{\hat{z}_0}{2} + \frac{1}{4\pi} \ln (1 - e^{-2\pi\hat{z}_0})^{-1} - \frac{1}{2\pi} \ln(2|\sin \lambda|). \quad (87)$$

Here $\lambda = \lambda_l(\hat{z}_0, w_0)$ (59).

I will address symmetry properties in the following section.

4.2. Symmetry properties

To any solution $(x, y)(u)$ there exists two solutions obtained by applying the symmetry γ and the time-reversible symmetry (A3) \mathcal{T}_τ :

$$\gamma(x, y, u)(t) = (-x, -y, u)(t), \quad \mathcal{T}_\tau(x, y, u)(t) = (x, -y, \tau - u)(-t).$$

Proposition 4. *Generic periodic orbits obtained from case (i) corresponds orbits that are symmetric with respect to γ and \mathcal{T}_τ ($4\pi\epsilon^{-1}$ -periodic) or just \mathcal{T}_τ ($2\pi\epsilon^{-1}$ -periodic). Case (ii) consists of non-symmetric $2\pi\epsilon^{-1}$ -periodic orbits or γ -symmetric $4\pi\epsilon^{-1}$ -periodic orbits.*

PROOF. Take $t = 0$ to be the time when $u = -\pi + \tau/2$. In case (i): $w_0 = 0, \pi/2 \bmod \pi$ up to an error $\mathcal{O}(\delta^{3/4} \ln \epsilon^{-1})$. This means that either $x(0) = 0$ or $y(0) = 0$ at $t = 0$. Consider a periodic orbit with $y(0) = 0$ first. This is the case when $\lambda_r = \lambda_l$ so the periodic orbit is a fix point of P . The initial condition is near $(x(0), 0, -\pi + \tau/2)$. The initial condition for the solution

$$\mathcal{T}_\tau(x, y, u) = (x, -y, \tau - u)(-t),$$

is near

$$(x(0), 0, \tau - (-\pi + \tau/2)) = (x(0), 0, \pi + \tau/2) = (x(0), 0, -\pi + \tau/2).$$

By local uniqueness of generic periodic orbits it follows that the periodic orbit coincides with its symmetry-related \mathcal{T}_τ -periodic orbit.

Consider next $x(0) = 0$. This is the case when $\lambda_r = \pi + \lambda_l$ so the periodic orbit is a fix point of P^2 . The initial condition is near $(0, y(0), -\pi + \tau/2)$. The image of P is near $(0, -y(0), -\pi + \tau/2)$. The initial condition for the solution $\gamma(x(t), y(t), u(t)) = (-x(t), -y(t), u(t))$ is near

$$(0, -y(0), -\pi + \tau/2).$$

Similarly the initial condition for the solution

$$\mathcal{T}_\tau(x(t), y(t), u(t)) = (x(-t), -y(-t), \tau - u(-t)),$$

is near

$$(0, -y(0), \tau - (-\pi + \tau/2)) = (0, -y(0), \pi + \tau/2) = (0, -y(0), -\pi + \tau/2).$$

By local uniqueness of generic periodic orbits it follows that the periodic orbit is γ and \mathcal{T}_τ -symmetric. The argument for the γ -symmetry can be modified so that it also applies to the $4\pi\epsilon^{-1}$ -periodic orbits in case (ii).

Corollary 3. *Generic symmetric periodic orbits from case (i) satisfy the \mathcal{T}_τ -symmetry condition*

$$(x, y, u)(t) = (x, -y, \tau - u)(-t),$$

when they are $2\pi\epsilon^{-1}$ -periodic, or a translated version

$$(x, y, u)(t + 2\pi\epsilon^{-1}) = (x, -y, \tau - u)(-t),$$

when they are $4\pi\epsilon^{-1}$ -periodic. In the latter case, the following γ -symmetry condition also applies

$$(x, y, u)(t + 2\pi\epsilon^{-1}) = (-x, -y, u)(t),$$

Note that these conditions imply that $y = 0$ at $u = \tau/2$ in agreement with the fact that (83) is just the condition that $\phi_l = 0 \bmod \pi$ (62).

I then show the following:

Theorem 14. *Consider $\hat{z}_0 \in [c_1^{-1}, c_2]$ and $\lambda_l(\hat{z}_0, w_0) \in [c^{-1}, \pi - c^{-1}] \cup [\pi + c^{-1}, 2\pi - c^{-1}]$. Then for both cases (i) (γ and \mathcal{T}_τ -symmetric if they have period $4\pi\epsilon^{-1}$ or just \mathcal{T}_τ if they have period $2\pi\epsilon^{-1}$) and (ii) (γ -symmetric or non-symmetric) the following statement holds true: For ϵ sufficiently small there exist $\mathcal{O}(\ln^2 \epsilon^{-1})$ -many unstable periodic orbits. The characteristic multipliers are $\mathcal{O}(\ln^{\pm 2} \epsilon^{-1})$.*

Remark 12. These orbits are unstable but cf. the estimate for the characteristic multipliers, the separation is much more modest in comparison with the typical exponential separation from the trivial periodic orbit $x = 0 = y$ (see also the appendix in [21]). This result gives 1° in our main result.

PROOF. *Case (i)*

I first consider case (i). Given that w_0 is already determined by (82), I am to solve the equation (83) for \hat{z}_0 . The dependency on \hat{z}_0 enters i.e. through $\hat{\rho}_0$ in (87) with $\lambda = \lambda_l$ as in (59). I need to exclude those $\lambda \notin (-c^{-1}, c^{-1}) \cup (\pi - c^{-1}, \pi + c^{-1})$ where $\text{tr } \partial P$ is near ± 2 . Following

$$\partial_{\hat{z}_0} \lambda_l = \ln \epsilon^{-1} + \mathcal{O}(1),$$

I realise that this gives rise to the exclusion of intervals within the interval $[c_1, c_2]$ of \hat{z}_0 -values which have widths of order $c^{-1} \ln^{-1} \epsilon^{-1}$. The complement contains closed intervals with widths of order $\ln^{-1} \epsilon^{-1}$. There are $\ln \epsilon^{-1}$ of such strips given an order 1 measure set of \hat{z}_0 -values. Now within each such strip for which $\lambda \notin (-c^{-1}, c^{-1}) \cup (\pi - c^{-1}, \pi + c^{-1})$ I have

$$(F_i^{(2)})'(\hat{z}_0) = -\frac{1}{\pi} \cot(\lambda) \ln^2(\epsilon^{-1}) + \mathcal{O}(\ln \epsilon^{-1}).$$

Given that $\cot \lambda \neq 0$ it therefore follows that there exist solutions of (83) within each strip. These solutions are separated by a distance of order $\ln^{-2} \epsilon^{-1}$. Furthermore, given that the order of the width of each strip is $\ln^{-1} \epsilon^{-1}$ there is an order of $\ln \epsilon^{-1}$ -many solutions within each strip. In total there are $\mathcal{O}(\ln^2 \epsilon^{-1})$ -many solutions as claimed.

Case (ii)

In this case I have to solve two equations: (85) and (86) for \hat{z}_0 and w_0 . Cf. (33) I can, however, solve for \hat{z}_0 and λ instead. Firstly

$$(F_{ii}^{(1)})'(\hat{z}_0) = 2 \ln(e_4 \epsilon^{-1}) + \mathcal{O}(1),$$

and so there exists an order $\ln \epsilon^{-1}$ many solutions \hat{z}_0 to (85). These are separated by a distance of order $\ln^{-1} \epsilon^{-1}$ within the order 1 set of \hat{z}_0 -values. For each solution to (85) I will solve (86) with respect to λ . Again I compute the derivative

$$\partial_\lambda F_{ii}^{(2)} = -\frac{1}{\pi} \cot(\lambda) \ln \epsilon^{-1} + \mathcal{O}(1),$$

using (87). Upon excluding a region around $\lambda = \pi/2$ and $3\pi/2$ where $\cot \lambda = 0$, the same reasoning, as used above, can be applied to conclude the existence of $\mathcal{O}(\ln \epsilon^{-1})$ -many solutions for each solution \hat{z}_0 of (85). In total there is therefore an order of $\ln^2 \epsilon^{-1}$ -many solutions.

The solutions are unstable cf. Lemma 13. From this also follows the estimates of the characteristic multipliers.

These solutions can all be continued into true solutions by applying the implicit function theorem taking ϵ sufficiently small.

From now I will focus on stable solutions.

4.3. Stable solutions

I focus on $\lambda = \lambda_l$ near $\pi/2$ and case (i) where $\lambda_r = \lambda_l \bmod \pi$. Case (ii) can be handled in a similar way. Following (78) I wish to consider λ of the form

$$\lambda = \frac{\pi}{2} + \hat{\lambda} \ln^{-1}(\epsilon^{-1}). \tag{88}$$

Lemma 15. *Let c be large and consider $\hat{z}_0 \neq \ln 2/(2\pi)$ and $\hat{z}_0 > c^{-1}$. Then for ϵ sufficiently small, all period orbits obtained from case (i) with λ_l as in (88) and where $\hat{\lambda}$ satisfies*

$$\hat{\lambda} \in \begin{cases} [-2\pi(2A + D|_{\lambda=\pi/2}) + c^{-1}, -c^{-1}] & \text{for } 2A + D|_{\lambda=\pi/2} > 0 \\ [c^{-1}, -2\pi(2A + D|_{\lambda=\pi/2}) - c^{-1}] & \text{for } 2A + D|_{\lambda=\pi/2} < 0 \end{cases}, \quad (89)$$

with

$$\begin{aligned} (2A + D|_{\lambda=\pi/2}) &= \frac{3e^{4\pi\hat{z}_0} - 6e^{2\pi\hat{z}_0} + 2}{(e^{2\pi\hat{z}_0} - 1)(e^{2\pi\hat{z}_0} - 2)} \\ &= 3 + 3e^{-2\pi\hat{z}_0} + \mathcal{O}(e^{-4\pi\hat{z}_0}), \end{aligned} \quad (90)$$

A and D as in (73) resp. (76), are stable.

Remark 13. The requirement $\hat{z}_0 \neq \ln 2/(2\pi)$ comes from the fact that $1 + p^2 \neq 0$ as $\arg(1 + p^2)$ appearing in (30) is not defined there. It also appears as a singularity of $C = \partial_{\hat{z}_0}(\arg(1 + p^2))$ and $D = \partial_{\lambda}\arg(1 + p^2)$ (76). In particular,

$$\lim_{\hat{z}_0 \rightarrow \ln 2/(2\pi)^{+/-}} (2A + D|_{\lambda=\pi/2}) = \pm\infty. \quad (91)$$

Remark 14. It is important to note, when comparing with the equations in [20, 21], the difference that our equations depend on the pseudo-angle λ (denoted by $\pi\eta$ in [20, 21]) strongly due to the factor $\ln \epsilon^{-1}$ of $\ln |\sin \lambda|$. Moreover, their γ_1 and γ_2 are both $\mathcal{O}(\ln \epsilon^{-1})$ in our case. Increasing γ_1 and γ_2 has the consequence of diminishing the stability region cf. Eq. (41) in [21] in agreement with Lemma 15.

PROOF. Eq. (88) is inserted into (74) which is then used in (70). This gives

$$\text{tr } \partial P = 2 + \frac{2\hat{\lambda}}{\pi} \left(\frac{\hat{\lambda}}{\pi} + 2A + D|_{\lambda=\frac{\pi}{2}} \right) + \mathcal{O}(\ln^{-1} \epsilon^{-1}).$$

A sufficient condition for stability is that $|\text{tr } \partial P| < 2$. Solving this inequality for $\hat{\lambda}$ one can then verify the statement about the stability for c large.

I will in the following prove 2° and 3° of the main theorem and therefore focus, as in 1°, on $\hat{z}_0 \in [c_1^{-1}, c_2]$.

To solve for stable orbits I proceed as above but this time I start by solving $\lambda_l(\hat{z}_0) = \lambda$ for \hat{z}_0 where λ_l is as in (59) with $w_0 = 0$, $\pi/2 \bmod \pi$ and the right hand side λ is as in (88). Given that

$$\partial_{\hat{z}_0} \lambda = \ln \epsilon^{-1} + \mathcal{O}(1),$$

this gives $\mathcal{O}(\ln \epsilon^{-1})$ -many intervals of lengths at least $\pi|2A+D|_{\lambda=\pi/2}| \ln^{-2} \epsilon^{-1}$. These intervals are separated by intervals of lengths $\mathcal{O}(\ln^{-1} \epsilon^{-1})$ where $|\text{tr } \partial P| > 2$. Also since

$$(F_i^{(2)})'(\hat{z}_0) = -\ln(\epsilon^{-1})(2A + D|_{\lambda=\pi/2}) + \mathcal{O}(1),$$

when $\lambda_l(\hat{z}_0)$ is as in (88), these intervals will under $F_i^{(2)}$ be mapped into $\mathcal{O}(\ln \epsilon^{-1})$ -many intervals in $\mathbb{R}/(\pi\mathbb{Z})$ of lengths $\pi|2A+D|_{\lambda=\pi/2}|^2 \ln^{-1} \epsilon^{-1}$. These intervals are separated by lengths

$$\pi(2A + D|_{\lambda=\pi/2}) + \mathcal{O}(\ln^{-1} \epsilon^{-1}). \quad (92)$$

Stable solutions are then the consequence of the intersection of these mapped intervals with 0 cf. (83). The stable solutions are therefore rare compared to the unstable ones: There can be at most $\ln \epsilon^{-1}$, in contrast to $\ln^2 \epsilon^{-1}$, but this is clearly very optimistic. Before supporting this claim by further analysis I first present some numerics for (7).

4.4. Numerics for (7)

In obtaining Fig. 3 I have used numerics to compute the number of stable solutions for (7) using the truncations of (83) and (89) for $\hat{z}_0 \in [0.1, 2]$ and 1000 different values of $\ln^{-1} \epsilon^{-1}$. The distribution of solutions are shown in Fig. 4. Out of the 1000 different values of $\ln^{-1} \epsilon^{-1}$ the case with no stable solutions occurred 368 times. In 632 of the cases I found at least one stable solution. Note that these *solutions* are solutions for the truncations. To continue them into true solutions one needs to invoke the implicit function theorem. For some of the extremely small values of ϵ considered it is difficult if *not* impossible to integrate the equations directly. Based on these observations, one could be led to the following conjecture:

Conjecture 1. *There exists an ϵ_0 and a number N so that the following holds true: For almost all $\epsilon \leq \epsilon_0$ the number of stable solutions is less than N .*

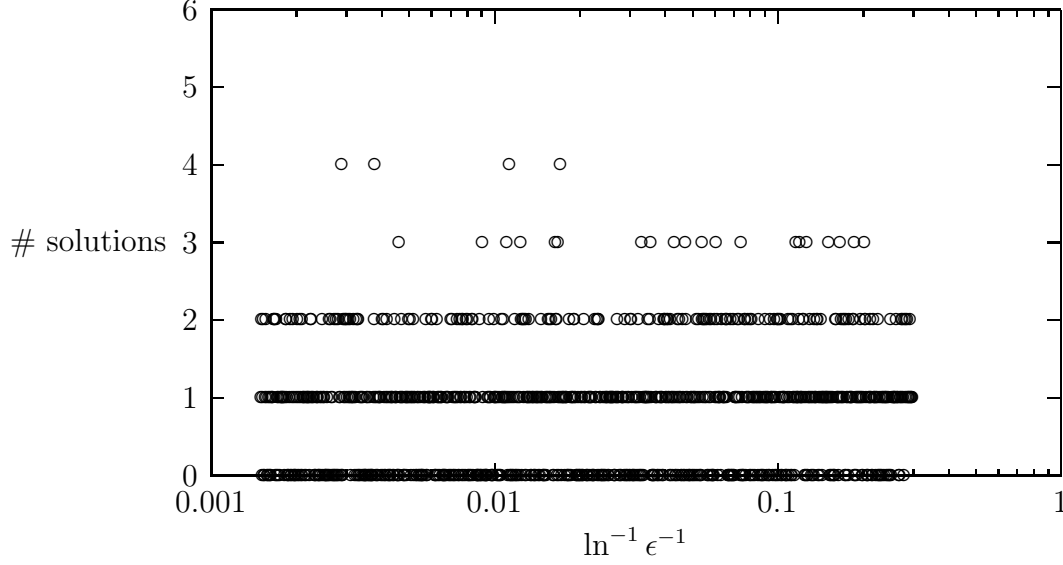


Figure 3: The number of stable solutions for $\hat{z}_0 \in [0.1, 2]$ for 1000 different values of $\ln^{-1} \epsilon^{-1}$ are shown with points.

In Table 1 I have documented the result from computing \mathcal{T}_π -symmetric, $2\pi\epsilon^{-1}$ -periodic solutions of (7). As in [21] I have used step-size control in the initial conditions $(x, 0)$ to carefully scan for periodic orbits in the interval $x \in (0, 0.5]$. I have used a 2-stage fully implicit Gauss-Legendre symplectic method for the time integration. POS in the second column of Table 1 is the number of periodic orbits. SPOS in the third column is for the number of periodic orbits. The fourth column gives the number of stable periodic orbits for $x \in (0, 2\epsilon^{1/2}]$. The final column gives the number of unstable periodic orbits for $x \in (0, 2\epsilon^{1/2}]$. The upper value $x = 2\epsilon^{1/2}$ corresponds to the upper bound $\hat{z}_0 = 2$ considered above. In agreement with the analysis and the computations above the stable periodic orbits are truly scarce within this interval. Also, in agreement with the results from [13, 20] the total number of periodic orbits, the number of stable period orbits, and the unstable ones all behave like ϵ^{-1} : The second and third column almost doubles when ϵ is halved. Finally, in agreement with my results the number of unstable periodic orbits within $x \in (0, 2\epsilon^{1/2}]$, the last column in Table 1, behaves like $\ln^2 \epsilon^{-1}$. A comparison is shown in Table 2.

ϵ	POS $x \in (0, 0.5]$	SPOS $x \in (0, 0.5]$	SPOS $x \in (0, 2\epsilon^{1/2}]$	UPOS $x \in (0, 2\epsilon^{1/2}]$
0.08	33	0	0	33
0.04	69	2	1	46
0.02	154	5	0	64
0.01	298	8	0	72
0.005	583	18	0	84
0.0025	1118	35	0	108

Table 1: Distribution of \mathcal{T}_π -symmetric, $2\pi\epsilon^{-1}$ -periodic orbits for (7).

ϵ	UPOS $x \in (0, 2\epsilon^{1/2}]$	UPOS-fit: $\lfloor 2.39 \ln^2 \epsilon^{-1} \rfloor + 21$	Relative Error
0.08	33	36	9.1%
0.04	46	45	2.2%
0.02	64	57	11%
0.01	72	71	1.4%
0.005	84	88	4.8%
0.0025	108	106	1.9%

Table 2: Comparison of the number of unstable periodic orbits in $x \in (0, 2\epsilon^{1/2}]$ with the number predicted by the theory (third column via linear fit).

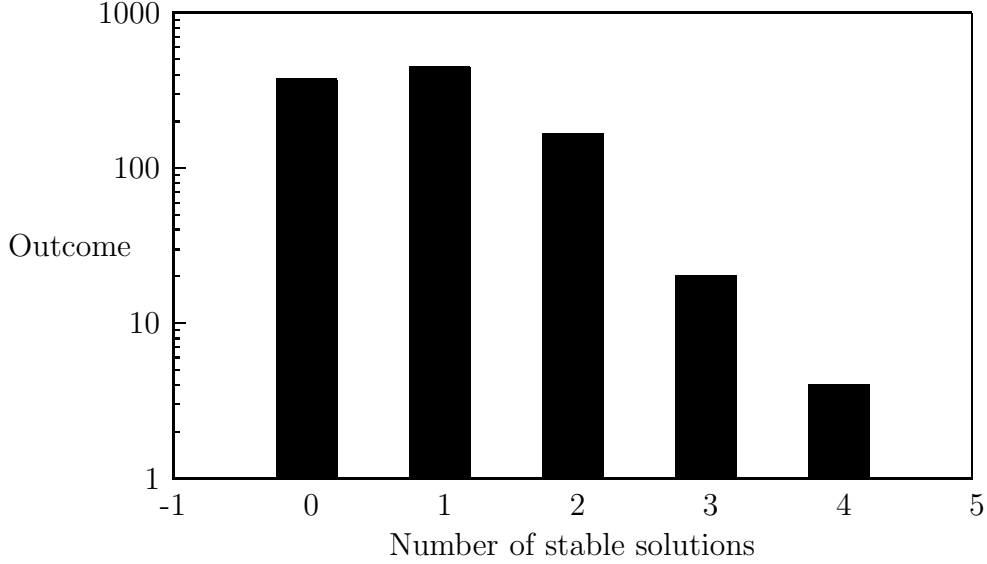


Figure 4: The number of outcomes for the different numbers of stable solutions. A total of 1000 different values of $\ln^{-1} \epsilon^{-1}$ were considered.

4.5. Part 2° of the main result

Suppose that an interval in $\mathbb{R}/(\pi\mathbb{Z})$, arising as the image under $F_i^{(2)}$ of a \hat{z}_0 -interval with $\lambda_l(\hat{z}_0)$ as in (88), intersects with 0. Then a stable solution exists. Let \hat{z}_0^0 denote the left end-point of the \hat{z}_0 -interval and denote by f_0 the image of \hat{z}_0 so that $f_0 = F_i^{(2)}(\hat{z}_0^0)$. The f_0 is an end-point of the mapped interval $\mathbb{R}/(\pi\mathbb{Z})$ since $(F_i^{(2)})'(\hat{z}_0) \neq 0$. The end-points of the following \hat{z}_0 -intervals are denoted by \hat{z}_0^n and similarly f_n will denote the image of \hat{z}_0^n under $F_i^{(2)}$. Here $n = 1, \dots, N$ with $N = \mathcal{O}(\ln \epsilon^{-1})$. This induces a mapping of the following form

$$\begin{aligned} f_{n+1} &= f_n + \varpi(\hat{z}_0^n) \mod \pi, \\ \hat{z}_0^{n+1} &= \hat{z}_0^n + \pi \ln^{-1} \epsilon^{-1} + \mathcal{O}(\ln^{-2} \epsilon^{-1}). \end{aligned} \tag{93}$$

I now try to work out an upper bound for how long one typically has to wait to obtain another solution. To obtain another solution, say \hat{z}_0^n , within the following N steps it is necessary for f_n to be within a distance of order $\ln^{-1} \epsilon^{-1}$ of f_0 . The answer depends on the arithmetic properties of the number $\varpi(\hat{z}_0^0)$. Consider therefore $c_1 > 0$ large and $d > 1$ and the following set of *Diophantine*

numbers

$$D_{c_1,d} = \{z \in [0,1) \mid |jz - i| \geq \frac{c_1^{-1}}{j^d}, i, j \in \mathbb{N}\}.$$

They have *almost* full measure: $1 - c_2(d)c_1^{-1}$ given that c_1 is large. Here the d dependency in $c_2(d)$ enters through a factor of $\sum_{j=1}^{\infty} j^{-d}$. This is why $d > 1$.

Proposition 5. *Suppose, with little loss of generality, that*

$$D_{c_1,d} \ni \pi^{-1}\varpi(\hat{z}_0^0) \bmod 1,$$

with c_1 small and $d > 1$. Then

$$|f_n - f_0| \gg \ln^{-1} \epsilon^{-1},$$

for all $n \leq \lfloor c_3^{-1} \ln^{(2+d)^{-1}} \epsilon^{-1} \rfloor$ with c_3 large.

PROOF. I linearise (93) about \hat{z}_0^0 and use the Diophantine property to obtain the following

$$|f_n - f_0| = |n\varpi(\hat{z}_0^0) + \mathcal{O}(n^2 \ln^{-1} \epsilon^{-1})| \geq \frac{c_1^{-1}}{\pi m^d} + \mathcal{O}(m^2 \ln^{-1} \epsilon^{-1}).$$

for all $n \leq m \ll \mathcal{O}(\ln \epsilon^{-1})$. Setting $m = \lfloor c_3^{-1} \ln^{(2+d)^{-1}} \epsilon^{-1} \rfloor$ then implies that $m^{-d} \gg m^2 \ln^{-1} \epsilon^{-1}$ provided c_3 is large enough. From here also follows that

$$|f_n - f_0| \geq \frac{c_1^{-1}}{2\pi m^d} \gg \ln^{-1} \epsilon^{-1},$$

for all such $n \leq m = \lfloor c_3^{-1} \ln^{(2+d)^{-1}} \epsilon^{-1} \rfloor$. This completes the proof.

The values \hat{z}_0^n are separated by the distance $\pi \ln^{-1} \epsilon^{-1} + \mathcal{O}(\ln^{-2} \epsilon^{-1})$. Given that one *typically* (in the sense that the diophantine numbers have almost full measure) have to wait longer than $m = \lfloor c_3^{-1} \ln^{(2+d)^{-1}} \epsilon^{-1} \rfloor$ steps between solutions (cf. Proposition 5), I can therefore, with little loss of generality, conclude that there can be at most

$$m^{-1} \ln \epsilon^{-1} = \mathcal{O}(\ln^{\frac{1+d}{2+d}} \epsilon^{-1}), \tag{94}$$

solutions within an order 1 interval of \hat{z}_0 -values. This gives 2° of the main result.

Remark 15. This result could perhaps be improved to something like Conjecture 1. It would require a theory for the asymptotic distribution of points on the circle $\mathbb{R}/(\pi\mathbb{Z})$ of “forced” circle maps of the form (93).

4.6. Distribution of stable solutions - Part 3° of the main result

Thus far I have kept ϵ fixed but small without being able to say anything about the existence or non-existence of stable solutions close to the bifurcating normally elliptic slow manifold. In fact, the numerics from above seem to indicate that both situations with existence and non-existence can occur. In this section, I will, however, study how stable solutions can be created when varying ϵ . This will cover part 3° of the main result. I still focus on case (i) although the result is also true for case (ii). Consider $\hat{z}_0^0 = \hat{z}_0^0(\epsilon)$ from above. It is a solution of $\lambda_l(\hat{z}_0^0) = \frac{\pi}{2} + \hat{\lambda} \ln^{-1} \epsilon^{-1}$ with $\hat{\lambda}$ help fixed at a value in the interior of the interval in (89). This gives the following expression for $(\hat{z}_0^0)'(\epsilon) = \frac{d}{d\epsilon} \hat{z}_0^0(\epsilon)$:

$$(\hat{z}_0^0)'(\epsilon) = -(\partial_{\hat{z}_0} \lambda_l)^{-1}(\partial_\epsilon \lambda_l + \mathcal{O}(\epsilon^{-1} \ln^{-2} \epsilon^{-1})) = \ln^{-1}(\epsilon^{-1}) \epsilon^{-2} e_3 + \mathcal{O}(\ln^{-2}(\epsilon^{-1}) \epsilon^{-2}).$$

The rate of change of $f_0 = f_0(\epsilon)$, which is the image of $\hat{z}_0^0(\epsilon)$ under $F_i^{(2)}$, is then also easily obtained:

$$f_0'(\epsilon) = ((2A + D|_{\lambda=\pi/2})e_3 - e_1)\epsilon^{-2} + \mathcal{O}(\ln^{-1}(\epsilon^{-1})\epsilon^{-2}). \quad (95)$$

Theorem 16. *Take any $\epsilon = \epsilon_1$ sufficiently small and set $I_1 = [\epsilon_1 - c_1 \epsilon_1^2, \epsilon_1 + c_1 \epsilon_1^2]$. Then within I_1 there will exist $\lfloor c_2^{-1} \ln \epsilon_1^{-1} \rfloor$ -many closed intervals of lengths $\geq c_3^{-1} \epsilon_1^2 \ln^{-1} \epsilon_1$ for which there exists stable solutions. Here c_1, c_2 and c_3 may be large but they can be taken to be independent of ϵ_1 .*

PROOF. Consider f_0 with derivative f_0' as in (95). Recall that f_0 is an end-point of a mapped interval that has length of order $\ln^{-1} \epsilon^{-1}$. If this interval does not intersect with 0 in $\mathbb{R}/(\pi\mathbb{Z})$ then one can simply, cf. (95), alter ϵ from ϵ_1 by an amount of $c_1 \epsilon_1^2$ to “push” this towards 0 (provided that $(2A + D|_{\lambda=\pi/2})e_3 - e_1 \neq 0$) and ensure the existence of a stable solution $(\hat{z}_0^0, \hat{\lambda})$. Since the mapped interval has a length of order $\ln^{-1} \epsilon_1$ this stable solution persists within a closed interval of ϵ -values with size of order $c_3^{-1} \epsilon_1^2 \ln^{-1} \epsilon_1^{-1}$. The \hat{z}_0 -interval based at \hat{z}_0^0 was arbitrary and the argument applies to all of the $\lfloor c_2^{-1} \ln \epsilon_1^{-1} \rfloor$ -many intervals with end-points at \hat{z}_0^n . The solutions can be continued into true solutions by applying the implicit function theorem.

By a similar argument to the one used in Proposition 5 it will also follow that if one takes $\lfloor c_4^{-1} \ln^{(2+d)^{-1}} \epsilon_1^{-1} \rfloor$ -following f_n 's, then they will typically be distant from each-other by a length of $\gg \ln^{-1} \epsilon_1^{-1}$. Here $d > 1$. The relative measure within I_1 of the union of the $\lfloor c_2^{-1} \ln \epsilon_1^{-1} \rfloor$ -many closed intervals of stable orbits will therefore typically be larger than $c_5^{-1} \ln^{-\frac{1+d}{2+d}} \epsilon_1^{-1} \gg \ln^{-1} \epsilon^{-1}$.

4.7. Part 4° of the main result

The last part 4° of the main result is the consequence of the following simple observation: The separation (92) between two consecutive mapped intervals having end-points e.g. at f_n and f_{n+1} are

$$3\pi + 3\pi e^{-2\pi\hat{z}_0} + \mathcal{O}(e^{-4\pi\hat{z}_0}) = 3\pi e^{-2\pi\hat{z}_0} + \mathcal{O}(e^{-4\pi\hat{z}_0}) \pmod{\pi},$$

cf. (90), dropping for simplicity the super-script on \hat{z}_0^n . By taking \hat{z}_0 “large” of size $\mathcal{O}(\ln \ln \epsilon^{-1})$ I can therefore guarantee that the separation is small being of order $\ln^{-1} \epsilon^{-1}$. Indeed, replace \hat{z}_0 by $\frac{1}{2\pi} \ln(\hat{Z}_0^{-1} \ln \epsilon^{-1})$. Then

$$\begin{aligned} 3\pi e^{-2\pi\hat{z}_0} + \mathcal{O}(e^{-4\pi\hat{z}_0}) &= 3\pi e^{\ln(\hat{Z}_0 \ln^{-1} \epsilon^{-1})} + \mathcal{O}(e^{\ln(\hat{Z}_0^2 \ln^{-2} \epsilon^{-1})}) \\ &= 3\pi \hat{Z}_0 \ln^{-1} \epsilon^{-1} + \mathcal{O}(\ln^{-2} \epsilon^{-1}). \end{aligned}$$

Taking \hat{z}_0 of this form implies, cf. e.g. (9), that the motion of (x, y) is *marginally* faster than the motion of u at $u = 0$.

All of the estimates above can be modified to account for \hat{z}_0 of this form. For P_i and P_o this follow immediately from the analysis. For P_{cr} one needs to analyse the correction term. This leads into similar calculations as the ones performed for P_i and P_o . The changes are therefore minor and I therefore leave the details out of the manuscript.

Now, remember that within an $\mathcal{O}(1)$ -interval of initial \hat{z}_0 -values there are $\ln \epsilon^{-1}$ many mapped intervals. Therefore by taking \hat{Z}_0 sufficiently small, but independent of ϵ , the union of these many intervals are guaranteed to cover the whole circle $\mathbb{R}/(\pi\mathbb{Z})$. In particular, there is one interval which intersects 0. This provides the existence of a stable solution and proves 4°.

Remark 16. Based on the numerics performed above, I have observed that the ratio between the value of x for the smallest stable period orbit of (7) and $\epsilon^{1/2}$ is slightly increasing with decreasing ϵ . It is, however, impossible to observe the $\ln^{1/2} \ln \epsilon^{-1}$ behaviour: $\ln \ln \epsilon^{-1} = c$ large requires an ϵ of the form e^{-e^c} !

5. Stability islands

Stability islands typically surround stable fix points of P or P^2 . How large are the stability islands? To address this I first need to introduce a further blow-up or scaling: $\hat{z}_0 = \ln^{-2}(\epsilon^{-1})\hat{Z}$, $w_0 = \ln^{-1}(\epsilon^{-1})\hat{W}$. The purpose

of this is to obtain an order 1 blow-up Poincaré mapping denoted by \hat{P} when λ_l is given as in (88). Indeed, from Lemma 12 it follows that the Jacobian of $\hat{P} = \hat{P}(\hat{Z}, \hat{W})$ with respect to these variables satisfy:

$$\partial_{(\hat{Z}, \hat{W})} \hat{P} = \begin{pmatrix} AD + \frac{(2A+D)\hat{\lambda}}{\pi} + \frac{\hat{\lambda}^2}{\pi^2} & -\frac{D\hat{\lambda}}{\pi} - \frac{\hat{\lambda}^2}{\pi^2} \\ -2A(2A+D) - \frac{(4A+D)\hat{\lambda}}{\pi} - \frac{\hat{\lambda}^2}{\pi^2} & 2 - AD + \frac{(2A+D)\hat{\lambda}}{\pi} + \frac{\hat{\lambda}^2}{\pi^2} \end{pmatrix} + \mathcal{O}(\ln^{-1} \epsilon^{-1}),$$

in case (i) and

$$\partial_{(\hat{Z}, \hat{W})} \hat{P} = \begin{pmatrix} AD - 2\frac{A\hat{\lambda}}{\pi} - \frac{\hat{\lambda}^2}{\pi^2} & \frac{\hat{\lambda}^2}{\pi^2} \\ -2a(2A+D) + \frac{\hat{\lambda}^2}{\pi^2} & 2 - AD + 2\frac{A\hat{\lambda}}{\pi} - \frac{\hat{\lambda}^2}{\pi^2} \end{pmatrix} + \mathcal{O}(\ln^{-1} \epsilon^{-1}),$$

in case (ii). Also

$$\hat{\lambda} = \pi/2 \ln \epsilon^{-1} - \hat{W} + (1 + \ln^{-1}(\epsilon^{-1}) \ln e_4) \hat{Z} + \ln(\epsilon^{-1}) G(\ln^{-2}(\epsilon^{-1}) \hat{Z}).$$

Note in particular how the trace in case (i) here agrees with the one presented in Lemma 15. Consider $(\hat{Z}, \hat{W}) = (\hat{Z}_e, \hat{W}_e)$ a stable fix points of \hat{P} or \hat{P}^2 . Typically KAM-theory can be applied to \hat{P} or \hat{P}^2 to conclude the existence of invariant curves surrounding $(\hat{Z}, \hat{W}) = (\hat{Z}_e, \hat{W}_e)$. The last one of such invariant curves creates a resonance island that measures $\mathcal{O}(1)$ in the (\hat{Z}, \hat{W}) -space; $\mathcal{O}(\ln^{-3} \epsilon^{-1})$ in (\hat{z}, w) -space; $\mathcal{O}(\epsilon \ln^{-3} \epsilon^{-1})$ in the original variables (x, y) .

6. Future work

The existence of the stable orbits in 4° provides a beginning of a connection with the work in [13, 20, 21]. I am currently working on making a more detailed connection to this work by providing a description of the distribution of periodic orbits further away from the slow manifold. This requires a combination of the techniques used here with those used in [13, 20, 21].

7. Acknowledgement

I would like to thank A. I. Neishtadt for pointing me in the direction of [19] and for suggestions leading to an improved manuscript.

Appendix A. Blow up for $u > 0$

In this appendix I present the derivation of (17) from (1). I first introduce

$$x = \pm\kappa(u) \pm \xi, \quad y = \pm\sigma, \quad (\text{A.1})$$

via the generating function:

$$G(x, \sigma, u, \nu) = \epsilon^{-1}u\nu \pm x\sigma - \kappa(u)\sigma,$$

and the equations

$$\xi = \partial_\sigma G, \quad y = \partial_x G, \quad v = \epsilon \partial_u G.$$

The transformation $(u, v, x, y) \mapsto (u, \nu, \xi, \sigma)$ transforms (1) into

$$\begin{aligned} H = & \nu - \frac{\epsilon(1 + \mathcal{O}(u))}{4\kappa(u)}\sigma + \frac{1}{2}(1 + \kappa(u)^2 M_0(u))\sigma^2 + \frac{1}{2}\kappa(u)^2 M_{02}(u)\sigma^4 \\ & + \kappa(u)M_0(u)(1 + \mathcal{O}(u))\xi\sigma^2 + \frac{1}{2}M_0(u)(1 + \mathcal{O}(u))\xi^2\sigma^2 \\ & + 2\kappa(u)^2(1 + 2\kappa(u)^2 P_0(u) + \kappa(u)^4 P_{01}(u))\xi^2 + 2\kappa(u)(1 + \mathcal{O}(u))\xi^3 \\ & + \frac{1}{2}(1 + \mathcal{O}(u))\xi^4 + \mathcal{O}(\xi\sigma^4 + \sigma^6 + \xi^3\sigma^2 + \xi^5), \\ \omega = & d\xi \wedge d\sigma + \epsilon^{-1}du \wedge d\nu, \end{aligned}$$

where $M_{01}(u) = \partial_{x^2}M(\kappa(u)^2, 0, u)$, $M_{011}(u) = \partial_{x^2}^2M(\kappa(u)^2, 0, u)$, $M_{02}(u) = \partial_{y^2}M(\kappa(u)^2, 0, u)$ and $P_0(u) = P_0(\kappa(u)^2, u)$, $P_{01}(u) = \partial_{x^2}P(\kappa(u)^2, u)$. I have here further translated ν to remove a term only depending on u . Moreover, I have used that

$$\kappa'(u) = \frac{1 + \mathcal{O}(u)}{4\kappa(u)},$$

which follows from assumption (A4). Next I set $\xi = \mu\delta^{3/4}\hat{\xi}$, $\sigma = \mu^2\delta^{3/4}\hat{\sigma}$, $\nu = \mu^4\delta^{3/2}\hat{\nu}$ and divide H by $\mu^4\delta^{3/2}$ to obtain a blow-up Hamiltonian system:

$$\begin{aligned} \hat{\Lambda} = & \hat{\nu} - \delta^{3/4}\frac{1 + \mathcal{O}(u)}{2\hat{\Omega}}\hat{\sigma} + \frac{1}{2}(1 + \kappa(u)^2 M_0(u))\hat{\sigma}^2 + \frac{1}{2}\mu^4\delta^{3/2}\kappa(u)^2 M_{02}(u)\hat{\sigma}^4 \\ & + \frac{1}{2}\mu^2\delta^{3/4}\hat{\Omega}(\hat{u})M_0(u)(1 + \mathcal{O}(u))\hat{\xi}\hat{\sigma}^2 + \frac{1}{2}\mu^2\delta^{3/2}M_0(u)(1 + \mathcal{O}(u))\hat{\xi}^2\hat{\sigma}^2 \\ & + \frac{1}{2}\hat{\Omega}(\hat{u})^2(1 + \kappa(u)^2 M_0(u))^{-1}\hat{\xi}^2 + \delta^{3/4}\hat{\Omega}(\hat{u})(1 + \mathcal{O}(u))\hat{\xi}^3 \\ & + \frac{1}{2}\delta^{3/2}(1 + \mathcal{O}(u))\hat{\xi}^4 + \mathcal{O}(\mu^5\delta^{9/4}\hat{\xi}\hat{\sigma}^4 + \mu^8\delta^3\hat{\sigma}^6 + \mu^3\delta^{9/4}\hat{\xi}^3\hat{\sigma}^2 + \mu\delta^{9/4}\hat{\xi}^5), \\ \hat{\omega} = & \mu^{-1}d\hat{\xi} \wedge d\hat{\sigma} + \mu^{-1}\delta^{-3/2}d\hat{u} \wedge d\hat{\nu} \end{aligned}$$

where $\hat{\Omega}$ is a scaled frequency given by

$$\hat{\Omega}(\hat{u})^2 = 4\mu^{-2}\kappa(u)^2(1 + 2\kappa(u)^2P_0(u) + \kappa(u)^4P_{01}(u))(1 + \kappa(u)^2M_0(u))^{-1} = 2\hat{u} + \mathcal{O}(\mu^2). \quad (\text{A.2})$$

All $\mathcal{O}(u)$ -terms have derivatives that are uniformly bounded on $u \geq 0$. To introduce action-angle coordinates I then first apply a transformation $(\hat{u}, \hat{\nu}, \hat{\xi}, \hat{\sigma}) \mapsto (\hat{u}, \hat{\nu}_0, \hat{\xi}_0, \hat{\sigma}_0)$ based on the following generating function:

$$G(\hat{\xi}, \hat{\sigma}_0, \hat{u}, \hat{\nu}_0) = \mu^{-1}\delta^{-3/2}\hat{u}\hat{\nu}_0 + \mu^{-1}\hat{\Omega}^{1/2}(1 + \kappa(u)^2M_0(u))^{-1/2}\hat{\xi}\hat{\sigma}_0,$$

and the equations

$$\begin{aligned} \hat{\xi}_0 &= \mu\partial_{\hat{\sigma}_0}G = \hat{\Omega}^{1/2}(1 + \kappa(u)^2M_0(u))^{-1/2}\hat{\xi}, \\ \hat{\sigma} &= \mu\partial_{\hat{\xi}}G = \hat{\Omega}^{1/2}(1 + \kappa(u)^2M_0(u))^{-1/2}\hat{\sigma}_0, \end{aligned}$$

and

$$\begin{aligned} \hat{\nu} &= \mu\delta^{3/2}\partial_{\hat{u}}G = \hat{\nu}_0 + \delta^{3/2}\hat{\Omega}(\hat{u})^{-2}(1 + \mathcal{O}(u))\hat{\xi}_0\hat{\sigma}_0 \\ &\quad - \frac{1}{2}\mu^2\delta^{3/2}((\kappa^2)'(u)M_0(u) + \kappa(u)^2M_0'(u))(1 + \mathcal{O}(u))\hat{\xi}_0\hat{\sigma}_0 \\ &= \hat{\nu}_0 + \delta^{3/2}\hat{\Omega}(\hat{u})^{-2}(1 + \mathcal{O}(u))\hat{\xi}_0\hat{\sigma}_0. \end{aligned} \quad (\text{A.3})$$

In the second to last equality I have used

$$\hat{\Omega}'(\hat{u}) = \frac{1 + \mathcal{O}(u)}{\hat{\Omega}(\hat{u})}, \quad (\text{A.4})$$

which follows from (A.2). Moreover, I have multiplied the last term by $1 = \hat{\Omega}^{-2}\hat{\Omega}^2$ and used that

$$\mu^2\hat{\Omega}(\hat{u})^2 = 2u(1 + \mathcal{O}(u)), \quad (\text{A.5})$$

to obtain the last equality (A.3). Here I have also used $\kappa(u)^2 = u/2(1 + \mathcal{O}(u))$ cf. (5) and (A4). The Hamiltonian takes the following form

$$\begin{aligned} \hat{\Lambda} &= \hat{\nu}_0 + \hat{\Omega}(\hat{u})\hat{\rho}_0 + \delta^{3/4}\hat{\Omega}(\hat{u})^{-1/2} \left((1 + \mathcal{O}(u))\hat{\xi}_0^3 - \frac{1}{2}(1 + \mathcal{O}(u))\hat{\sigma}_0 \right) \\ &\quad + \delta^{3/2}\hat{\Omega}(\hat{u})^{-2} \left(\frac{1}{2}(1 + \mathcal{O}(u))\hat{\xi}_0^4 + (1 + \mathcal{O}(u))\hat{\xi}_0\hat{\sigma}_0 \right) \\ &\quad + \frac{1}{2}\mu^4\delta^{3/2}\kappa(u)^2M_{02}(u)\hat{\Omega}(\hat{u})^2\hat{\sigma}_0^4 + \frac{1}{2}\mu^2\delta^{3/4}\hat{\Omega}(\hat{u})^{3/2}M_0(u)(1 + \mathcal{O}(u))\hat{\xi}_0\hat{\sigma}_0^2 \\ &\quad + \frac{1}{2}\mu^2\delta^{3/2}M_0(u)(1 + \mathcal{O}(u))\hat{\xi}_0^2\hat{\sigma}_0^2 \\ &\quad + \mathcal{O}(\mu^5\delta^{9/4}\hat{\Omega}^{3/2}\hat{\xi}_0\hat{\sigma}_0^4 + \mu^8\delta^3\hat{\Omega}^3\hat{\sigma}_0^6 + \mu^3\delta^{9/4}\hat{\Omega}^{-1/2}\hat{\xi}_0^3\hat{\sigma}_0^2 + \mu\delta^{9/4}\hat{\Omega}^{-5/2}\hat{\xi}_0^5). \end{aligned}$$

The action-angle variables $(\hat{\varrho}_0, \tilde{\phi}_0)$ are the symplectic polar coordinates of $(\hat{\xi}_0, \hat{\sigma}_0)$:

$$\hat{\xi}_0 = \sqrt{2\hat{\varrho}_0} \cos \tilde{\phi}_0, \quad \hat{\sigma}_0 = \sqrt{2\hat{\varrho}_0} \sin \tilde{\phi}_0.$$

As $u < 0$ I also here wish to write the Hamiltonian function $\hat{\Lambda}$ only in terms of δ and the scaled frequency $\hat{\Omega}$. For this I use (A.5). In total:

$$\begin{aligned} \hat{\Lambda} = & \hat{\nu}_0 + \hat{\Omega}(\hat{u})\hat{\varrho}_0 + \delta^{3/4}\hat{\Omega}(\hat{u})^{-1/2} \left((1 + \mathcal{O}(u))\hat{\xi}_0^3 - \frac{1}{2}(1 + \mathcal{O}(u))\hat{\sigma}_0 + uM_0(u)(1 + \mathcal{O}(u))\hat{\xi}_0\hat{\sigma}_0^2 \right) \\ & + \delta^{3/2}\hat{\Omega}(\hat{u})^{-2} \left(\frac{1}{2}(1 + \mathcal{O}(u))\hat{\xi}_0^4 + (1 + \mathcal{O}(u))\hat{\xi}_0\hat{\sigma}_0 + u^3M_{02}(u)(1 + \mathcal{O}(u))\hat{\sigma}_0^4 \right. \\ & \left. + uM_0(u)(1 + \mathcal{O}(u))\hat{\xi}_0^2\hat{\sigma}_0^2 \right) + \mathcal{O}(\hat{\Omega}^{-7/2}\delta^{9/4} + \hat{\Omega}^{-5}\delta^3). \end{aligned}$$

Appendix B. Approximation of P_o

I start by presenting two lemmata similar to Lemma 5 and Lemma 6:

Lemma 17. *Let q and p be positive real numbers satisfying $0 < q < p$. Then there exists a constant $c = c(\hat{u}_*)$ so that*

$$\hat{F}(\hat{u})^{-2p} \leq c^{p-q} \hat{F}(\hat{u})^{-2q},$$

for all $\hat{u} \in [-\mu^{-2}(\pi - \tau/2), -\hat{u}_*]$.

Lemma 18. *Let $q \in \overline{\mathbb{R}}_+$. Given an integrable function $r = r(\hat{u})$ satisfying the following estimate*

$$|r(\hat{u})| \leq \hat{F}(\hat{u})^{-2q}, \quad \hat{u} \in [\hat{u}_*, \mu^{-2}\tau/2].$$

If $q < 1$ then there exists a $c_1 = c_1(q)$ so that

$$\left| \int_{-\mu^{-2}(\pi-\tau/2)}^{-\hat{u}_*} r(\hat{u}) d\hat{u} \right| \leq c_1 \mu^{-2(1-q)}.$$

If $q = 1$ then there exists a c_2 so that

$$\left| \int_{-\mu^{-2}(\pi-\tau/2)}^{-\hat{u}_*} r(\hat{u}) d\hat{u} \right| \leq c_2 \ln(\mu^{-2}\hat{u}_*^{-1}).$$

Finally if $q > 1$ then the corresponding integral is uniformly bounded with respect to ϵ : There exists a c_3 so that

$$\left| \int_{-\mu^{-2}(\pi-\tau/2)}^{-\hat{u}_*} r(\hat{u}) d\hat{u} \right| \leq c_3 (q-1)^{-1} \hat{u}_*^{1-q}.$$

The proofs of these lemmata are almost identical to the proofs of Lemma 5 and Lemma 6 and therefore left out.

I then recall the form of \hat{H} in (14):

$$\hat{H} = h_0(\hat{u}, \hat{v}, \hat{z}_0) + r_0(\hat{u}, \hat{v}, \hat{z}_0, w_0) + \mathcal{O}(\hat{F}(\hat{u})^{-5} \delta^3), \quad (\text{B.1})$$

and note that

$$\begin{aligned} \bar{r}_0 &= \frac{1}{2\pi} \int_0^{2\pi} r_0(\hat{u}, \hat{z}_0, \tau) d\tau = \frac{3}{4} \delta^{3/2} \hat{F}(\hat{u})^{-2} \hat{z}_0^2 - \frac{1}{4} \delta^{3/2} \hat{F}(\hat{u})^{-2} f(u) M_0(u) \hat{z}_0^2 \\ &= \frac{3}{4} \delta^{3/2} \hat{F}(\hat{u})^{-2} (1 + \mathcal{O}(u)) \hat{z}_0^2. \end{aligned}$$

I then introduce the following generating function

$$\begin{aligned} G(\hat{u}, \hat{v}, \hat{z}_0, w_1) &= \delta^{-3/2} \hat{u} \hat{v} + \hat{z}_0 w_1 + \hat{F}(\hat{u})^{-1} \int_0^{\tilde{\phi}_1} \tilde{r}_0(\hat{u}, \hat{z}_0, \tau) d\tau, \\ \tilde{r}_0 &= r_0 - \bar{r}_0. \end{aligned}$$

This generates a symplectic transformation $(\hat{u}, \hat{v}_0, \hat{z}_0, w_0) \mapsto (\hat{u}, \hat{v}, \hat{z}, w)$ with $\hat{z}_0 = \hat{z} + \mathcal{O}(\delta^{3/2})$ transforming H (B.1) into

$$\hat{H} = \hat{v} + \hat{F}(\hat{u}) \hat{z} + \frac{3}{4} \delta^{3/2} \hat{F}(\hat{u})^{-2} (1 + \mathcal{O}(u)) \hat{z}^2 + \mathcal{O}(\hat{F}(\hat{u})^{-5} \delta^3).$$

The equations of motion are

$$\begin{aligned} \frac{d\hat{z}}{d\hat{u}} &= \mathcal{O}(\hat{F}(\hat{u})^{-5} \delta^{3/2}), \\ \frac{dw}{d\hat{u}} &= -\delta^{-3/2} \hat{F}(\hat{u}) - \frac{3}{2} \hat{F}(\hat{u})^{-2} (1 + \mathcal{O}(u)) \hat{z} + \mathcal{O}(\hat{F}(\hat{u})^{-5} \delta^{3/2}). \end{aligned} \quad (\text{B.2})$$

In accordance with the definition of P_o I consider \hat{u} here from $\hat{u} = -\mu^{-2}(\pi - \tau/2)$ to $\hat{u} = -\hat{u}_*$.

Lemma 19. \hat{z}_0 is conserved on the interval from $\hat{u} = -\mu^{-2}(\pi - \tau/2)$ to $\hat{u} = -\hat{u}_*$ up to an error of order $\mathcal{O}(\delta^{3/2})$.

PROOF. I can take $p = 5/2 > 1$ in Lemma 18 to control the variation of \hat{z} by an error of order $\delta^{3/2}$. Since $\hat{z}_0 = \hat{z} + \mathcal{O}(\delta^{3/2})$ the result follows.

By a similar argument, I estimate the effect of the remainder in (B.2) by $\mathcal{O}(\delta^{3/2})$ and I compute the variation in the angle w by

$$\begin{aligned} w(-\hat{u}_*) &= w(-\mu^{-2}(\pi - \tau/2)) - \int_{-\mu^{-2}(\pi - \tau/2)}^{-\hat{u}_*} \left(\delta^{-3/2} \hat{F}(\hat{u}) + \frac{3}{2} \hat{F}(\hat{u})^{-2} (1 + \mathcal{O}(u)) \hat{z}_0 + \mathcal{O}(\hat{F}(\hat{u})^{-2} \delta^{3/2}) \right) d\hat{u} \\ &\quad + \mathcal{O}(\delta^{3/2}). \end{aligned} \tag{B.3}$$

As above in Section 3.1, the remainder $\mathcal{O}(\hat{F}(\hat{u})^{-2} \delta^{3/2})$ in the integral comes from $\hat{z}(\hat{u}) = \hat{z}_0 + \mathcal{O}(\delta^{3/2})$, with $\hat{z}_0 = \text{const.}$ on this interval. This can be estimated from above by a term of order $\delta^{3/2} \ln \mu^{-1}$ using $q = 1/2$ in Lemma 18. The following lemma gives asymptotics of the two other integrals appearing in (B.3).

Lemma 20.

$$\delta^{-3/2} \int_{-\mu^{-2}(\pi - \tau/2)}^{-\hat{u}_*} \hat{F}(\hat{u}) d\hat{u} = \mu^{-3} \delta^{-3/2} e_3 - \frac{2}{3} \delta^{-3/2} \hat{u}_*^{3/2} + \mathcal{O}(\epsilon^{2/3} \delta^{-5/2}).$$

with

$$e_3 = \int_0^{\pi - \tau/2} (-f(-u))^{1/2} du.$$

Moreover, there exists some positive constant e_4 such that

$$\int_{-\mu^{-2}(\pi - \tau/2)}^{-\hat{u}_*} \hat{F}(\hat{u})^{-2} d\hat{u} = \ln(e_4 \mu^{-2} \delta^{-1}) - \ln(\delta^{-1} \hat{u}_*) + \mathcal{O}(\mu^2).$$

PROOF. I use (11):

$$\begin{aligned} \delta^{-3/2} \int_{-\mu^{-2}(\pi - \tau/2)}^{-\hat{u}_*} \hat{F}(\hat{u}) d\hat{u} &= \mu^{-3} \delta^{-3/2} \int_{u_*}^{\pi - \tau/2} (-f(-u))^{1/2} du \\ &= \mu^{-3} \delta^{-3/2} e_3 - \mu^{-3} \delta^{-3/2} \int_0^1 (-f(-u_* s))^{1/2} ds u_*, \end{aligned} \tag{B.4}$$

here $u_* = \mu^2 \hat{u}_*$. Since $f(u) = u + \mathcal{O}(u^2)$ cf. (A4) for small u , I can for the last integral in (B.4) use the same argument used for (51) to complete the first part of the proof.

For the second part:

$$\int_{-\mu^{-2}(\pi-\tau/2)}^{-\hat{u}_*} \hat{F}(\hat{u})^{-2}(1 + \mathcal{O}(u))d\hat{u} = \int_{u_*}^{\pi-\tau/2} (-f(-u))^{-1}(1 + \mathcal{O}(u))du.$$

I write $(-f(-u))^{-1} = u^{-1} + \mathcal{O}(1)$ for u small and complete the result as in the proof of Lemma 10.

Following this lemma I can therefore write (49) as

$$\begin{aligned} w_0(-\hat{u}_*) &= w_0(-\mu^{-2}(\pi - \tau/2)) - \epsilon^{-1}e_3 - \frac{3}{2}\ln(e_4\mu^{-2}\delta^{-1})\hat{z}_0 + \frac{2}{3}\delta^{-3/2}\hat{u}_*^{3/2} + \frac{3}{2}\ln(\delta^{-1}\hat{u}_*)\hat{z}_0 \\ &\quad + \mathcal{O}(\delta^{3/2}\ln\mu^{-1}). \end{aligned} \tag{B.5}$$

using here that $w_0 = w + \mathcal{O}(\delta^{3/2})$. Colleting the results, I obtain Proposition 2.

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